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Asymptotics of a class of multidimensional Laplace-type integrals. I. Double integrals

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Contents

By representing double Laplace-type integrals as iterated Mellin–Barnes integrals, followed by judicious application of residue theory, the authors obtain new asymptotic expansions of integrals of the form $\int_0^\infty \int_0^\infty e^{-\lambda f(x,y)} g(x,y)\, dx dy$ for a large class of 'phases', f, and 'amplitudes', g. The allowed phases are 'polynomials' (non-integer powers are permitted) with an isolated, though possibly degenerate, critical point at the origin. The determination of which residues to use in constructing the expansions is characterized in elementary geometric terms. Numerical examples highlighting the use of the expansions are supplied, as is a discussion of the relationship between the geometry of the Newton diagram of the phase and the asymptotic scales used in the expansions.

Keywords: asymptotic expansions; iterated Mellin–Barnes integrals; Newton diagram; single critical point

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1. Introduction

In this paper, and its sequel, we are concerned with the task of developing asymptotic expansions of n-dimensional Laplace-type integrals

$$I(\lambda)=\int_0^\infty\cdots\int_0^\infty \mathrm{e}^{-\lambda f(x_1,x_2,\ldots,x_n)}g(x_1,x_2,\ldots,x_n)\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n, \tag{1.1}$$

as the parameter $\lambda\to\infty$. Initially, we shall assume $g\equiv 1$ and f is a 'polynomial' (we allow non-integer powers) with positive coefficients, although later in the paper, we show how some restrictions may be relaxed. To ensure the existence of (1.1), we will assume f contains the terms $x_1^{\mu_1}+x_2^{\mu_2}+\cdots+x_n^{\mu_n}$.

Integrals of the type (1.1), and the equivalent oscillatory form, have been studied by a variety of authors, and much is known about the problem of determining the large-λ behaviour of this integral under a range of hypotheses for the case of $n=1$. For multidimensional Laplace integrals, the problem is considerably more difficult, with most work to date being of a qualitative character under very broad assumptions. For very general f and g (real analytic f, smooth g), work by Malgrange (1974) produced a structure theorem for the form the asymptotic expansion of $I(\lambda)$ takes. The proof relied on the use of a resolution of the singularity of the phase function, f, at the origin, and was not amenable to computation.

Subsequent work by Arnold *et al.* (1988) provides some sharper results, including estimates of the order of the leading term expressed in terms of the remoteness of the Newton diagram of the phase, f, (see the following section for a definition of this term), but again, because their work relies on algebraic/topological methods (principally, the resolution of singularities), little is said about the problem of constructing full expansions of $I(\lambda)$. Related work by Vasil'ev (1977) obtains the leading term of the expansion and the correct order in terms of the remoteness of the Newton diagram, but does so in a fashion unsuitable for obtaining higher order approximations.

Other efforts have been made to deduce the asymptotic behaviour of $I(\lambda)$ using information contained in the Newton diagram of f. One novel approach, by Denef & Sargos (1989), works with dual structures of the faces of the Newton diagram and sidesteps the use of a resolution of singularities by resorting instead to a 'dissection' of the integration domain, with different changes of variables (determined by faces of the Newton diagram) brought to bear over different pieces of the integration domain. Denef & Sargos obtain more detailed information about the poles of the distribution $f_+^{\mathrm{s}}\equiv\max(f^{\mathrm{s}},0)$ whose action on test functions is given by $\varphi\mapsto\int_{\boldsymbol{R}^n}f_+^{\mathrm{s}}(x_1,\ldots,x_n)\varphi(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n$. The style of argument employed suggests that each face in the Newton diagram ought to appear in some sense in the asymptotics of f_+^{s}. Their result can be rendered in the form of an exponential or oscillatory integral by passing to a Mellin transform, after the fashion employed by Wong & McClure (1981, p. 518).

A more elementary approach was employed by Dostal & Gaveau (1987, 1989) where an argument employing a rescaling of the polynomial phase f, which we examine in this paper, for each face in the Newton polygon led to an asymptotic approximation, but with coefficients that were expressed as integrals of exponential functions resulting from their rescaling operations.

If one is willing to cast aside the desire to use the geometric content of the Newton diagram of f, then the results are still quite limited. In short, one can represent the multiple integral $I(\lambda)$ as the Fourier or Laplace transform of a lower dimensional integral as is done in Jones & Kline (1958) or Wong & McClure (1981), deduce the asymptotics of the function defined by the lower dimensional integral and subsequently apply Watson's lemma or a similar argument. This approach, however, becomes very unwieldy when the phase function has something other than a non-degenerate Hessian matrix at the origin.

Our approach to the problem completely avoids the use of a resolution of singularity, and makes no attempt to represent the integral as an integral transform of a function defined by an integral over a lower dimensional object. As such, we can avoid the analytical difficulties encountered in the approaches mentioned above, and provide a great deal of information relating the geometry of the Newton diagram of the phase, f, to the asymptotics of (1.1). Indeed, the geometry of the Newton diagram surfaces in the course of developing the expansions of (1.1), and we shall find that $I(\lambda)$ has a compound expansion comprising one series per face of the Newton diagram.

We are, however, limited to those phases, f, that have a single critical point at the origin, and nowhere else in the domain of integration of (1.1). The degeneracy of the critical point at the origin plays no role whatsoever in our analysis.

After a brief summary of terminology and some preliminary estimates used in the remainder of the paper, we turn to a Mellin–Barnes integral representation of (1.1).

2. Definitions and some useful estimates

For a real analytic function f, with Maclaurin series

$$f(x,y)=\sum_{m,n\geqslant 0} a_{mn}x^m y^n,$$

define the *carrier* of f to be the set of ordered pairs of non-negative integers $\{(m,n): a_{mn}\neq 0\}$; for a polynomial f, this is just the set of powers of monomials comprising f.

For each point P in the carrier of f, consider the positive quadrant $\boldsymbol{R}^2_+=\{(x,y): x>0, y>0\}$ translated by P so that the origin is sited at P. Call this translate of the positive quadrant $P+\boldsymbol{R}^2_+$. Form the union of these translated quadrants and then its convex hull. The boundary of the convex hull will be composed of two rays parallel to the coordinate axes, as well as a polygonal path composed of finitely many line segments. This polygonal path is termed the *Newton diagram* or *Newton polygon* of f. Figure 1 displays the two-dimensional Newton diagram for the polynomial $2x^5+x^3y^2+xy^2+3y^5$.

This definition of the Newton diagram, from Brieskorn & Knörrer (1986), can easily be extended to functions f with arbitrary non-negative real powers for the monomials comprising f, an extension which we adopt throughout this paper. Extension to functions of three or more variables is also straightforward: one replaces the use of the positive quadrant in the construction by the use of the positive orthant of appropriate dimension.

Given a Newton diagram for a function $f(x,y)$, we form the ray issuing from the origin with direction vector $\boldsymbol{e}=(1,1)$. Because the Newton diagram meets both coordinate axes, there is a number, d say, for which $d\boldsymbol{e}$ is a point of the Newton diagram. The number d is the *distance* to the Newton diagram, and the quantity $-1/d$ is termed the *remoteness* of the Newton diagram. The points (m,n) used to construct the Newton diagram we term *internal points* if they do not lie on coordinate axes. If these points lie on the Newton diagram, they are termed *vertices*.

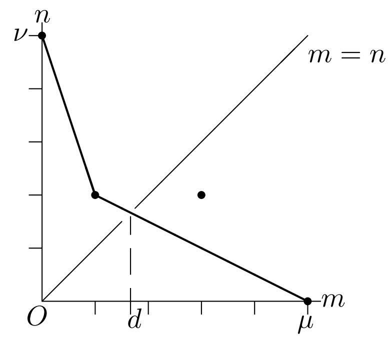

Figure 1. Newton diagram for $2x^5 + x^3y^2 + xy^2 + 3y^5$ showing the line $m = n$ and the distance d to the Newton diagram.

We will be estimating the magnitude of the integrands of Mellin–Barnes integrals, so the following consequences of Stirling's formula, presented in Paris & Wood (1986, § 2.1.3), will prove to be useful. Let $s = \rho e^{i\theta}$ where ρ and θ are real. Then, for $\beta > 0$ and α complex, we have

$$\left.\begin{aligned}\log|\Gamma(\alpha+\beta s)| &\sim \beta\rho\cos\theta\log\beta\rho - \beta\rho(\theta\sin\theta+\cos\theta)\\ &\quad +(\mathrm{Re}(\alpha)-\tfrac{1}{2})\log\beta\rho\\ \log|\Gamma(\alpha-\beta s)| &\sim -\beta\rho\cos\theta\log\beta\rho + \beta\rho(\theta\sin\theta+\cos\theta)\\ &\quad +(\mathrm{Re}(\alpha)-\tfrac{1}{2})\log\beta\rho - \log|\sin\pi(\alpha-\beta s)|,\end{aligned}\right\} \tag{2.1}$$

as $\rho \to \infty$, provided $|\arg(\alpha \pm \beta s)| < \pi$.

A number of expressions appear frequently throughout our analysis for which a more compact notation will prove convenient:

$$\left.\begin{aligned}K &= 1+\mu k, \quad L = 1+\mu l, \quad R = 1+\mu r,\\ K' &= 1+\nu k, \quad L' = 1+\nu l, \quad R' = 1+\nu r.\end{aligned}\right\} \tag{2.2}$$

These definitions are brought to the attention of the reader when first used.

3. Representation as a Mellin–Barnes integral

For double integrals (higher dimensional integrals will be treated in a sequel), we can, under the hypotheses of § 1, write $I(\lambda)$ as

$$I(\lambda) = \int_0^\infty\int_0^\infty \exp\left[-\lambda\left(x^\mu + \sum_{p=1}^{k} c_p x^{m_p} y^{n_p} + y^\nu\right)\right] dx dy, \tag{3.1}$$

where all exponents are positive and all constants c_p are positive real numbers (or complex with positive real part). Let us define for each $r = 1, 2, \ldots, k$

$$\delta_r = 1 - \frac{m_r}{\mu} - \frac{n_r}{\nu}, \tag{3.2}$$

and set

$$\left.\begin{aligned}\boldsymbol{m} &= (m_1, m_2, \ldots, m_k), \quad \boldsymbol{t} = (t_1, t_2, \ldots, t_k),\\ \boldsymbol{n} &= (n_1, n_2, \ldots, n_k), \quad \boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k).\end{aligned}\right\} \tag{3.3}$$

Apply the integral representation, with the contour indented to the right of the origin,

$$e^{-z} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(t) z^{-t}\, dt, \quad |\arg z| < \tfrac{1}{2}\pi, \quad z \neq 0, \tag{3.4}$$

to each factor $\exp(-\lambda c_r x^{m_r} y^{n_r})$ in the integrand of (3.1) so that we can recast the integral in (3.1), after interchanging the order of integration, as

$$I(\lambda) = \frac{\lambda^{1/\mu - 1/\nu}}{\mu\nu}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^k \Gamma(\boldsymbol{t})\Gamma\left(\frac{1-\boldsymbol{m}\cdot\boldsymbol{t}}{\mu}\right)\Gamma\left(\frac{1-\boldsymbol{n}\cdot\boldsymbol{t}}{\nu}\right)\boldsymbol{c}^{-\boldsymbol{t}}\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\, d\boldsymbol{t}, \tag{3.5}$$

where we have set

$$\Gamma(\boldsymbol{t}) = \Gamma(t_1)\Gamma(t_2)\cdots\Gamma(t_k), \quad \boldsymbol{c}^{-\boldsymbol{t}} = c_1^{-t_1}c_2^{-t_2}\cdots c_k^{-t_k} \quad \text{and} \quad d\boldsymbol{t} = dt_1 dt_2\cdots dt_k.$$

The 'dot' appearing between two vector quantities is just the usual Euclidean dot product. The integration contours are indented to the right away from the origin, to avoid the pole of the integrand present there.

By considering each integral in (3.5) separately, for example,

$$J_r = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(t_r)\Gamma\left(\frac{1-\boldsymbol{m}\cdot\boldsymbol{t}}{\mu}\right)\Gamma\left(\frac{1-\boldsymbol{n}\cdot\boldsymbol{t}}{\nu}\right)c_r^{-t_r}\lambda^{-\delta_r t_r}\, dt_r,$$

we can determine, reasoning as in Paris & Wood (1986, § 2.1.3, and rule 1 of p. 21), the sector in the complex λ-plane in which the integral J_r converges. By (2.1), we have for $t_r = \rho e^{i\theta}$, $\rho \to \infty$,

$$\log\left|\Gamma(t_r)\Gamma\left(\frac{1-\boldsymbol{m}\cdot\boldsymbol{t}}{\mu}\right)\Gamma\left(\frac{1-\boldsymbol{n}\cdot\boldsymbol{t}}{\nu}\right)c_r^{-t_r}\lambda^{-\delta_r t_r}\right| \sim \delta_r\rho\cos\theta\log\rho + A\rho + B\log\rho, \tag{3.6}$$

where

$$A = -\delta_r(\theta\sin\theta + \cos\theta) - \cos\theta(\delta_r\log|\lambda| + \log|c_r|) + \sin\theta(\delta_r\arg\lambda + \arg c_r)$$
$$\qquad - \left(\frac{m_r}{\mu} + \frac{n_r}{\nu}\right)\pi|\sin\theta| - \cos\theta\cdot\log\left[\left(\frac{m_r}{\mu}\right)^{m_r/\mu}\left(\frac{n_r}{\nu}\right)^{n_r/\nu}\right]$$

$$B = \operatorname{Re}\left\{\frac{1}{\mu}(1 - \boldsymbol{m}^*\cdot\boldsymbol{t}^*) + \frac{1}{\nu}(1 - \boldsymbol{n}^*\cdot\boldsymbol{t}^*)\right\} - \tfrac{3}{2};$$

here, we temporarily allow c_r to be complex, and write $\boldsymbol{m}^*$, $\boldsymbol{n}^*$ and $\boldsymbol{t}^*$ for those quantities in (3.3) with the rth entry deleted. Since the integral defining J_r is taken over the imaginary axis (except near the origin), we set $\theta = \pm\frac{1}{2}\pi$ to find the estimate for the logarithm of the dominant real part of the integrand given by

$$-\delta_r\tfrac{1}{2}\pi\rho - \left(\frac{m_r}{\mu} + \frac{n_r}{\nu}\right)\pi\rho \pm \rho(\delta_r\arg\lambda + \arg c_r)$$
$$= -\tfrac{1}{2}\pi\rho\left(1 + \frac{m_r}{\mu} + \frac{n_r}{\nu}\right) \pm \rho(\delta_r\arg\lambda + \arg c_r),$$

for large ρ. The integrand of J_r decays exponentially if this estimate tends to $-\infty$ as $\rho \to \infty$ and hence, treating the cases of positive and negative δ_r separately, shows that absolute convergence is assured if

$$\left|\arg\lambda + \frac{1}{\delta_r}\arg c_r\right| < \tfrac{1}{2}\pi\frac{1 + m_r/\mu + n_r/\nu}{|\delta_r|}; \tag{3.7}$$

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the convergence of the integral J_r in the case of $\delta_r = 0$ (when J_r is independent of λ) is handled similarly. Evidently, the factor following $\frac{1}{2}\pi$ in the right-hand side of (3.7) is greater than unity, so that each integral J_r defines an analytic function of λ in a sector including the imaginary λ-axis when c_r is positive.

Before developing expansions, we make the remark that our work also applies to integrals of the form

$$\tilde{I}(\lambda) = \int_0^\infty \int_0^\infty \exp\left[-\lambda \left(x^\mu + \sum_{p=1}^{k} c_p x^{m_p} y^{n_p} + y^\nu \right) \right] x^\alpha y^\beta \, \mathrm{d}x \mathrm{d}y,$$

where $\alpha > -1$, $\beta > -1$. To see this, we observe that $\tilde{I}$ may be transformed into an integral of the form (3.1) upon application of the simple change of variables $x = X^p$, $y = Y^q$. The differential $x^\alpha y^\beta \, \mathrm{d}x \mathrm{d}y$ then becomes $pqX^{p-1+\alpha p} Y^{q-1+\beta q} \, \mathrm{d}X \mathrm{d}Y$, whence the choice $p = 1/(1+\alpha)$, $q = 1/(1+\beta)$ removes the powers of X and Y from the differential. The result is an integral of the type (3.1), albeit with different powers appearing in the phase.

4. Asymptotics with one internal point

We begin our analysis by examining the special case $k = 1$ in (3.1) and (3.5), namely

$$\begin{aligned} I(\lambda) &= \int_0^\infty \int_0^\infty \exp[-\lambda(x^\mu + x^{m_1} y^{n_1} + y^\nu)] \, \mathrm{d}x \mathrm{d}y \\ &= \frac{\lambda^{-1/\mu - 1/\nu}}{\mu\nu} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t) \Gamma\left(\frac{1 - m_1 t}{\mu}\right) \Gamma\left(\frac{1 - n_1 t}{\nu}\right) \lambda^{-\delta_1 t} \, \mathrm{d}t; \end{aligned} \qquad (4.1)$$

the constant c_1 in (3.1) has been removed by rescaling integration variables and the large parameter λ. This is a simple one-dimensional contour integral, and its analysis poses no difficulty. The asymptotics of (4.1) are obtained in conventional fashion by displacing the integration contour to the left or right in the complex t-plane, according as the dominant real part of the logarithm of the integrand tends to plus or minus infinity, respectively (see Wong 1989, ch. III).

In the integrand of (4.1) let us set $t = \rho \mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$. By employing the estimates (2.1) as in (3.6), we find that the dominant real part of the logarithm of the integrand of (4.1) behaves as

$$\rho \cos\theta \log\rho \left(1 - \frac{m_1}{\mu} - \frac{n_1}{\nu}\right) = \delta_1 \rho \cos\theta \log\rho$$

for large ρ (recall (3.2)). This tends to plus or minus infinity according as $\delta_1 > 0$ or $\delta_1 < 0$. If δ_1 vanishes, we must conduct our analysis with finer estimates.

The sign of δ_1 has geometric significance. To see this, we note that the back face for the Newton diagram has equation

$$\frac{m}{\mu} + \frac{n}{\nu} - 1 = 0, \qquad (4.2)$$

the equation of the line joining the points $(\mu, 0)$ and $(0, \nu)$. If a point (m, n) lies on the same side of the line (4.2) as the origin, then (m, n) must satisfy $m/\mu + n/\nu - 1 < 0$. Thus, the statement that $\delta_1 > 0$ is equivalent to the statement that $P_1 = (m_1, n_1)$

lies below the back face (i.e. on the same side of (4.2) as the origin). The Newton diagram in this case has two faces: a line segment joining $(\mu, 0)$ to P_1, and a line segment joining P_1 to $(0, \nu)$ (see figure 2). If δ_1 vanishes, then P_1 must lie on the back face, and if $\delta_1 < 0$, then P_1 lies behind the back face (i.e. on the side of (4.2) not containing the origin).

(*a*) *Vertex P_1 in front of the back face*

In this setting, we deduce that the dominant real part of the logarithm of the integrand (3.6) tends to $+\infty$, from which we conclude that contributions to the asymptotic behaviour of (4.1) result from poles obtained by displacing the integration contour to the right (see Slater 1966). Candidate poles arise in two sequences: one from poles of $\Gamma((1-m_1t)/\mu)$, and one from poles of $\Gamma((1-n_1t)/\nu)$. For the present, we will assume that these two sequences share no points, so that all such poles are simple.

The poles of $\Gamma((1-m_1t)/\mu)$ are easily seen to occur at the points

$$t^{(1)} = (1 + \mu k)/m_1 = K/m_1, \quad k = 0, 1, 2, \ldots, \tag{4.3}$$

while poles of $\Gamma((1-n_1t)/\nu)$ occur at the points

$$t^{(2)} = (1 + \nu k)/n_1 = K'/n_1, \quad k = 0, 1, 2, \ldots; \tag{4.4}$$

here, use has been made of the notation presented in (2.2). Poles from the $t^{(1)}$ sequence give rise to the formal asymptotic series (suppressing the factor $\lambda^{-1/\mu-1/\nu}$ appearing in (4.1))

$$I_1 = \frac{1}{m_1\nu} \sum_k \frac{(-1)^k}{k!} \Gamma\left(\frac{K}{m_1}\right) \Gamma\left(\frac{m_1 - n_1K}{m_1\nu}\right) \lambda^{-\delta_1 K/m_1}, \tag{4.5}$$

whereas poles from the $t^{(2)}$ sequence give rise to asymptotic series (again, suppressing the power of λ appearing as a leading factor in (4.1))

$$I_2 = \frac{1}{\mu n_1} \sum_k \frac{(-1)^k}{k!} \Gamma\left(\frac{K'}{n_1}\right) \Gamma\left(\frac{n_1 - m_1K'}{\mu n_1}\right) \lambda^{-\delta_1 K'/n_1}; \tag{4.6}$$

in each of the formal asymptotic sums constituting I_1 and I_2, the index k ranges over all non-negative integers.

Upon assembling these two asymptotic series, we obtain the asymptotic expansion, for $\lambda \to \infty$,

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}(I_1 + I_2), \tag{4.7}$$

where $I(\lambda)$, I_1 and I_2 are given in (4.1), (4.5) and (4.6), respectively.

(*b*) *Vertex P_1 on or behind the back face*

We saw earlier that $\delta_1 < 0$ implies that the point P_1 lies above the back face of the Newton diagram. Indeed, the Newton diagram in this case is composed only of the back face joining $(\mu, 0)$ to $(0, \nu)$. The dominant real part of the logarithm of the integrand of (4.1) clearly tends to $-\infty$, so we must displace the integration contour in (4.1) to the left in order to obtain the asymptotics of $I(\lambda)$.

Doing so, we find that the only poles appearing in our evaluations are those occurring at the non-negative integers, $t = -k$, $k = 0, 1, 2, \ldots$. The expansion of $I(\lambda)$ is then immediate:

$$I(\lambda) \sim \frac{\lambda^{-1/\mu-1/\nu}}{\mu\nu} \sum_k \frac{(-1)^k}{k!} \Gamma\left(\frac{1+m_1 k}{\mu}\right) \Gamma\left(\frac{1+n_1 k}{\nu}\right) \lambda^{\delta_1 k}, \tag{4.8}$$

for $\lambda \to \infty$. We observe that this is precisely the expansion that results from (4.1) by developing $\exp(-\lambda x^{m_1} y^{n_1})$ into its Maclaurin series in x and y, followed by termwise integration.

We also note that this series is the convergent series expansion for (4.1) that results in the case $\delta_1 > 0$ when the integration contour is displaced to the left, and can be used to compute $I(\lambda)$ for modest values of λ for $\delta_1 > 0$. Convergence in this case follows from a straightforward application of the ratio test and the well-known asymptotic behaviour of the ratio of Γ functions (Olver 1974, § 5.1) which shows that the ratio of consecutive terms in (4.8) has the large-k behaviour

$$\frac{1}{k+1}\left(\frac{m_1 k}{\mu}\right)^{m_1/\mu}\left(\frac{n_1 k}{\nu}\right)^{n_1/\nu}|\lambda|^{\delta_1} \sim \left(\frac{m_1}{\mu}\right)^{m_1/\mu}\left(\frac{n_1}{\nu}\right)^{n_1/\nu} k^{-\delta_1}|\lambda|^{\delta_1}. \tag{4.9}$$

However, if $\delta_1 \geqslant 0$, then we must have $m_1 < \mu$ and $n_1 < \nu$, so all factors in the above ratio are less than unity. Absolute convergence of the series (4.8) in the case $\delta_1 \geqslant 0$ follows.

The case of $\delta_1 = 0$ corresponds to the situation where P_1 is located on the back face, between the points $(\mu, 0)$ and $(0, \nu)$. Because $\delta_1 = 0$, the integral (4.1) no longer has λ appearing in the integrand, so we need only evaluate the integral. The result, a convergent series,

$$\begin{aligned} I_0 &\equiv \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t)\Gamma\left(\frac{1-m_1 t}{\mu}\right)\Gamma\left(\frac{1-n_1 t}{\nu}\right) \mathrm{d}t \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(\frac{1+m_1 k}{\mu}\right)\Gamma\left(\frac{1+n_1 k}{\nu}\right), \end{aligned}$$

gives the evaluation

$$I(\lambda) = \frac{\lambda^{-1/\mu-1/\nu}}{\mu\nu} I_0. \tag{4.10}$$

Convergence of the series representation of I_0 follows from the discussion surrounding (4.9).

(c) Remoteness and order of leading terms

We demonstrate here the interesting relationship that exists between the order of the leading terms in the expansions (4.7), (4.8) and (4.10), and the remoteness of the Newton diagram: the order of the leading term in the expansion of $I(\lambda)$ is equal to the remoteness of the Newton diagram of the phase (cf. Vasil'ev 1977).

Suppose that P_1 lies below the 45° line in the mn-plane, i.e. that $m_1 > n_1$, and that $\delta_1 > 0$. (A similar argument can be brought to bear for the case where P_1 lies above the 45° line, i.e. $n_1 > m_1$.) Let L_1 and L_2 be the lines through $(0, \nu)$ and P_1, and P_1 and $(\mu, 0)$, respectively. Let m_1^* and m_2^* be the abscissas of the points of intersection of L_1 and L_2 with the 45° line, respectively (see figure 2).

Evidently, we must have $m_2^* < m_1^*$, so $-1/m_2^* < -1/m_1^*$. Elementary analytical geometry furnishes us with $m_1^* = \nu m_1/(m_1 + \nu - n_1)$ and $m_2^* = \mu n_1/(\mu + n_1 - m_1)$.

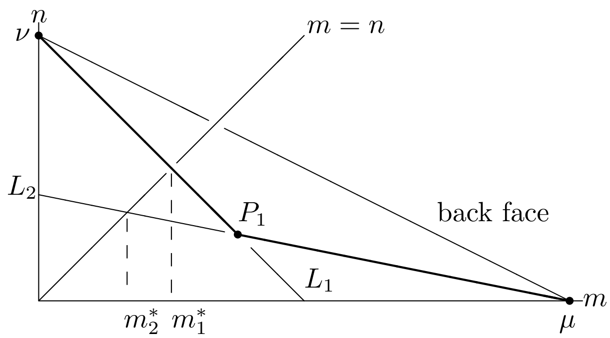

Figure 2. Geometry of the Newton diagram for the case $\delta_1 > 0$.

From (4.7), the leading terms in the expansion of $I(\lambda)$ for the case $\delta_1 > 0$ give rise to

$$I(\lambda) \sim \frac{\lambda^{-1/\mu-1/\nu-\delta_1/m_1}}{m_1\nu}\Gamma\left(\frac{1}{m_1}\right)\Gamma\left(\frac{m_1-n_1}{m_1\nu}\right) + \frac{\lambda^{-1/\mu-1/\nu-\delta_1/n_1}}{\mu n_1}\Gamma\left(\frac{1}{n_1}\right)\Gamma\left(\frac{n_1-m_1}{\mu n_1}\right).$$

The first term has order

$$-\frac{1}{\mu}-\frac{1}{\nu}-\frac{\delta_1}{m_1} = -\frac{1}{\nu}+\frac{n_1}{m_1\nu}-\frac{1}{m_1} = -\frac{1}{m_1^*},$$

and the second,

$$-\frac{1}{\mu}-\frac{1}{\nu}-\frac{\delta_1}{n_1} = -\frac{1}{\mu}+\frac{m_1}{n_1\mu}-\frac{1}{n_1} = -\frac{1}{m_2^*}.$$

In view of the inequality $-1/m_2^* < -1/m_1^*$, we see that the dominant term in the expansion (4.7) must come from the term of order $-1/m_1^*$, i.e. from the term whose order is the same as the remoteness of the Newton diagram.

If, now, $\delta_1 \leqslant 0$, P_1 lies on or behind the back face (the Newton diagram in these cases) and the remoteness is then just $-1/\mu - 1/\nu$. This is the same as the order of the leading term in (4.8) or the order of the single term in (4.10).

Of special interest is the case where P_1 lies on the 45° line and in front of the back face. In this case, $m_1 = n_1$, and from examination of (4.3) and (4.4), it is apparent that whatever the choices of μ and ν, the leading term in the expansion of $I(\lambda)$ will contain a factor of $\log\lambda$, stemming from the presence of a double pole in the Mellin–Barnes representation (4.1). This situation is explored in detail in § 7.

5. Asymptotics with two internal points

With $k = 2$ in (3.1) and (3.5), our integral assumes the form

$$I(\lambda) = \frac{\lambda^{-1/\mu-1/\nu}}{\mu\nu}\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{1-\boldsymbol{m}\cdot\boldsymbol{t}}{\mu}\right) \times\Gamma\left(\frac{1-\boldsymbol{n}\cdot\boldsymbol{t}}{\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1\mathrm{d}t_2; \qquad (5.1)$$

we are using the vector representations in (3.3) with $k = 2$, and the constants c_1 and c_2 in (3.1) have been set, for ease of discussion, to unity. It is a simple matter to restore the constants (see § 6 *b*, for example). We assume throughout this section that all poles of the integrand are simple.

With two internal points present, say $P_1 = (m_1, n_1)$ and $P_2 = (m_2, n_2)$, an additional possible shape of the Newton diagram arises: we can have: (a) both internal points acting as vertices of the Newton diagram, so that the diagram is formed from three non-collinear line segments; (b) both internal points on the Newton diagram, but with two of the three line segments collinear; (c) one point only on the Newton diagram, with the other internal point behind any of the line segment 'faces' of the Newton diagram; or finally, (d) both internal points could lie on or behind the back face joining $(\mu, 0)$ to $(0, \nu)$, so that the Newton diagram is just the back face. We shall consider the asymptotics of $I(\lambda)$ that arise in each of the cases that keep at least one internal point in front of the back face, and in the course of our analysis, two principles become apparent: first, the compound asymptotic expansions that result will be formed from series associated with each of the faces of the Newton diagram; second, the presence of internal points lying behind the Newton diagram do not play any role in the leading terms of the expansions.

Before proceeding further in our investigations, let us impose some structure on our points P_1 and P_2: we shall assume that $\mu > m_1 > m_2$, $n_1 < n_2 < \nu$, so that the quantities $M \equiv m_1 - m_2$ and $N \equiv n_2 - n_1$ are positive.

(*a*) *The convex case*

By the convex case, we mean that both P_1 and P_2 are vertices on the Newton diagram, and that all three line segments comprising the diagram are non-collinear (i.e. no two line segments are collinear). In this setting, we shall find it convenient to introduce the quantity

$$\Delta \equiv m_1 n_2 - m_2 n_1. \tag{5.2}$$

Observe that Δ is the signed area of the parallelogram generated by the vectors $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, in that order, where $\boldsymbol{P}$ is the position vector defined by P. Since the ordering imposed on P_1 and P_2 gives $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ a positive orientation, the quantity Δ must be positive.

Additionally, elementary analytic geometry reveals that the line through P_1 and P_2 cuts the m- and n-axes at Δ/N and Δ/M, respectively. By considering the intersection points of lines through $(\mu, 0)$ and P_1, and P_2 and $(0, \nu)$, with the coordinate axes, we also find

$$\frac{\nu m_2 N}{\nu - n_2} < \Delta < \mu N \quad \text{and} \quad \frac{\mu n_1 M}{\mu - m_1} < \Delta < \nu M; \tag{5.3}$$

the intersection of the line through $(0, \nu)$ and P_2 with the m-axis is indicated as (1_m) in figure 3; that of the line through P_1 and P_2 with the m-axis labelled (2_m) in the figure, and so on. The labelled intersection points (1_m), (2_m), (1_n) and (2_n) correspond to $m = m_2\nu/(\nu - n_2)$, $m = \Delta/N$, $n = \mu n_1/(\mu - m_1)$ amd $n = \Delta/M$, respectively.

Let us begin by displacing the t_2 contour in (5.1) first. With $t_2 = \rho e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$, we see that the logarithm of the modulus of the integrand of (5.1) exhibits the large ρ behaviour

$$\delta_2 \rho \cos\theta \log\rho + \mathcal{O}(\rho).$$

Since P_2 is in front of the back face (cf. equation (4.2)), the quantity δ_2 must be positive from which it follows that the order estimate above tends to $+\infty$. Thus,

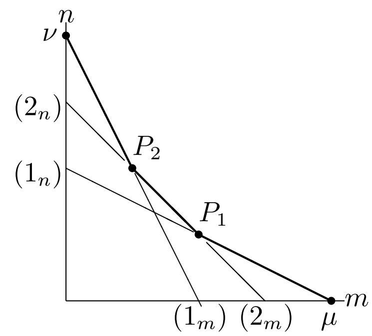

Figure 3. Newton diagram for two internal points in the 'convex' case. The labels (1_m), (2_m), (1_n) and (2_n) are described in the text.

we displace the t_2 contour to the right to pick up contributions to the asymptotics of $I(\lambda)$. Candidate poles arise in two sequences (here assumed to share no points in common) determined by the poles of $\Gamma((1-m_1t_1-m_2t_2)/\mu)$ and $\Gamma((1-n_1t_1-n_2t_2)/\nu)$.

The poles of the first of these Γ-functions are located at

$$t_2^{(1)} = (K - m_1t_1)/m_2, \quad k = 0, 1, 2, \ldots, \tag{5.4}$$

while those of the second occur at

$$t_2^{(2)} = (K' - n_1t_1)/n_2, \quad k = 0, 1, 2, \ldots \tag{5.5}$$

(cf. (4.3) and (4.4)). Poles from the $t_2^{(1)}$ sequence give rise to the formal sum (omitting the leading factor of $\lambda^{-1/\mu-1/\nu}$)

$$I_1 = \frac{1}{m_2\nu}\sum_k \frac{(-1)^k}{k!}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1)\Gamma(t_2^{(1)})\Gamma\left(\frac{1-n_1t_1-n_2t_2^{(1)}}{\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \tag{5.6}$$

with $\boldsymbol{\delta}\cdot\boldsymbol{t} = \delta_1t_1 + \delta_2t_2^{(1)}$, and those from the $t_2^{(2)}$ sequence,

$$I_2 = \frac{1}{\mu n_2}\sum_k \frac{(-1)^k}{k!}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1)\Gamma(t_2^{(2)})\Gamma\left(\frac{1-m_1t_1-m_2t_2^{(2)}}{\mu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \tag{5.7}$$

with $\boldsymbol{\delta}\cdot\boldsymbol{t} = \delta_1t_1 + \delta_2t_2^{(2)}$.

The integrals in the I_1 series have integrands which reduce to

$$\Gamma(t_1)\Gamma\left(\frac{K-m_1t_1}{m_2}\right)\Gamma\left(\frac{m_2-n_2K+\Delta t_1}{m_2\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

from which we find, with $t_1 = \rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta| < \frac{1}{2}\pi$, has the logarithm of the modulus behaving for large ρ as

$$\rho\cos\theta\log\rho(\Delta - \nu M)/m_2\nu + \mathcal{O}(\rho). \tag{5.8}$$

From (5.3), we see that the factor following $\log\rho$ is negative, whence the logarithm of the modulus of the integrand tends to $-\infty$ as $\rho\to\infty$. We must therefore displace the t_1 contour in each integral in I_1 to the left. We find that there are three sequences of candidate poles arising: one each from the Γ-functions appearing in the integrands in I_1.

The factor $\Gamma(t_1)$ has its poles at $t_1^{(1)} = -l$, $l = 0, 1, 2, \ldots$, whereas the second factor, $\Gamma(t_2^{(1)}) = \Gamma((K - m_1 t_1)/m_2)$ would have its poles at points $(K + m_2 l)/m_1$, l a non-negative integer. However, it is clear that these points are positive, for all non-negative integral values of k and l, so no poles of this factor are encountered in displacing the t_1 contour to the left. The final sequence of candidate poles arises from poles of $\Gamma((1 - n_1 t_1 - n_2 t_2^{(1)})/\nu) = \Gamma((m_2 - n_2 K + \Delta t_1)/m_2 \nu)$, given by

$$t_1^{(2)} = \frac{n_2}{\Delta} K - \frac{m_2}{\Delta}(1 + \nu l) = \frac{n_2 K - m_2 L'}{\Delta}, \tag{5.9}$$

for non-negative integral k and l satisfying $t_1^{(2)} \leqslant 0$. Here, we have set $1 + \nu l \equiv L'$ (recall (2.2)).

The $t_1^{(1)}$ sequence of poles in I_1 gives rise to the formal asymptotic series (again, suppressing the factor $\lambda^{-1/\mu - 1/\nu}$)

$$\begin{aligned} I_{11} &= \frac{1}{m_2 \nu} \sum_{k,l} \frac{(-1)^{k+l}}{k!l!} \Gamma(t_2^{(1)}(t_1^{(1)})) \Gamma\left(\frac{1 - n_1 t_1^{(1)} - n_2 t_2^{(1)}(t_1^{(1)})}{\nu}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}} \\ &= \frac{1}{m_2 \nu} \sum_{k,l} f_{11}(k, l), \end{aligned} \tag{5.10}$$

where

$$f_{11}(k, l) \equiv \frac{(-1)^{k+l}}{k!l!} \Gamma\left(\frac{K + m_1 l}{m_2}\right) \Gamma\left(\frac{m_2 - n_2 K - \Delta l}{m_2 \nu}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \tag{5.11}$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1 t_1^{(1)} - \delta_2 t_2^{(1)}(t_1^{(1)}) = -\frac{\delta_2}{m_2} K + \frac{\Delta - \nu M}{m_2 \nu} l. \tag{5.12}$$

The $t_1^{(2)}$ sequence of poles in I_1 generates the formal asymptotic series (with suppressed leading factor of $\lambda^{-1/\mu - 1/\nu}$)

$$I_{12} = \frac{1}{\Delta} \sum_{k,l'} \frac{(-1)^{k+l}}{k!l!} \Gamma(t_1^{(2)}) \Gamma(t_2^{(1)}(t_1^{(2)})) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}} = \frac{1}{\Delta} \sum_{k,l'} f_{12}(k, l), \tag{5.13}$$

where

$$f_{12}(k, l) \equiv \frac{(-1)^{k+l}}{k!l!} \Gamma\left(\frac{n_2 K - m_2 L'}{\Delta}\right) \Gamma\left(\frac{m_1 L' - n_1 K}{\Delta}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \tag{5.14}$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1 t_1^{(2)} - \delta_2 t_2^{(1)}(t_1^{(2)}) = \frac{\Delta - \mu N}{\mu \Delta} K + \frac{\Delta - \nu M}{\nu \Delta} L'. \tag{5.15}$$

The prime attached to the summation index l appearing in (5.13) indicates that the index is subject to the restriction imposed earlier on (5.9), namely that $t_1^{(2)} \leqslant 0$.

We repeat the analysis for the integrals appearing in the series I_2. The integrals in (5.7) have integrands which reduce to

$$\Gamma(t_1) \Gamma\left(\frac{K' - n_1 t_1}{n_2}\right) \Gamma\left(\frac{n_2 - m_2 K' - \Delta t_1}{\mu n_2}\right) \lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

which has, with $t_1 = \rho e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$, the logarithm of the modulus behaving for large ρ as

$$\rho \cos\theta \log \rho (\mu N - \Delta)/\mu n_2 + \mathcal{O}(\rho).$$

From (5.3), we see that the factor following $\log\rho$ is positive whence the logarithm of the modulus of the integrand tends to $+\infty$ as $\rho\to\infty$. Accordingly, we must displace the contour to the right to pick up poles contributing to the asymptotics. We find that there are two sequences of relevant poles: the first, from poles of $\Gamma((K'-n_1t_1)/n_2)$, given by

$$t_1^{(1)}=(K'+n_2l)/n_1; \tag{5.16}$$

the second, from poles of $\Gamma((n_2-m_2K'-\Delta t_1)/\mu n_2)$, given by

$$t_1^{(2)}=\frac{n_2}{\Delta}(1+\mu l)-\frac{m_2K'}{\Delta}=\frac{n_2L-m_2K'}{\Delta}, \tag{5.17}$$

where we have set $1+\mu l\equiv L$ (see (2.2)). In (5.16) and (5.17), the parameter l is a non-negative integer, but in the case of $t_1^{(2)}$, l is subject to the additional constraint $t_1^{(2)}>0$.

The $t_1^{(1)}$ sequence of poles in (5.16) gives rise to the formal asymptotic series (with omission of the λ factor as before)

$$\begin{aligned}I_{21}&=\frac{1}{\mu n_1}\sum_{k,l}\frac{(-1)^{k+l}}{k!l!}\Gamma(t_2^{(2)}(t_1^{(1)}))\Gamma\left(\frac{1-m_1t_1^{(1)}-m_2t_2^{(2)}(t_1^{(1)})}{\mu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\\&=\frac{1}{\mu n_1}\sum_{k,l}f_{21}(k,l),\end{aligned} \tag{5.18}$$

where

$$f_{21}(k,l)\equiv\frac{(-1)^{k+l}}{k!l!}\Gamma\left(\frac{K'+n_2l}{n_1}\right)\Gamma\left(\frac{n_1-m_1K'-\Delta l}{\mu n_1}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \tag{5.19}$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t}=-\delta_1t_1^{(1)}-\delta_2t_2^{(2)}(t_1^{(1)})=-\frac{\delta_1}{n_1}K'+\frac{\Delta-\mu N}{\mu n_1}l. \tag{5.20}$$

The $t_1^{(2)}$ sequence gives rise to

$$I_{22}=\frac{1}{\Delta}\sum_{k,l'}\frac{(-1)^{k+l}}{k!l!}\Gamma(t_1^{(2)})\Gamma(t_2^{(2)}(t_1^{(2)}))\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}=\frac{1}{\Delta}\sum_{k,l'}f_{12}(l,k), \tag{5.21}$$

where the prime attached to l indicates that it is subject to the restriction $t_1^{(2)}>0$.

Collecting together the expansions, we find that

$$I(\lambda)\sim\lambda^{-1/\mu-1/\nu}(I_{11}+I_{12}+I_{21}+I_{22}),$$

for large λ. However, we note that, if we relabel the summation indices in I_{22} by putting $k\to l$ and $l\to k$, then the linear restriction $t_1^{(2)}=n_2L/\Delta-m_2K'/\Delta>0$ becomes the constraint $n_2K/\Delta-m_2L'/\Delta>0$, exactly the complement of the constraint $t_1^{(2)}=n_2K/\Delta-m_2L'/\Delta\leqslant 0$ which governs I_{12}. Thus, the two series I_{12} and I_{22} may be fused into a single series without constraint (save that of k and l being non-negative integers), to give the final asymptotic form

$$I(\lambda)\sim\lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_2\nu}\sum_{k,l}f_{11}(k,l)+\frac{1}{\Delta}\sum_{k,l}f_{12}(k,l)+\frac{1}{\mu n_1}\sum_{k,l}f_{21}(k,l)\right\}, \tag{5.22}$$

for $\lambda\to\infty$. Observe that the first series associates in a natural way with the face of the Newton diagram joining the vertices $(0,\nu)$ and P_2, the second with the face joining P_1 and P_2, and the third with the face joining $(\mu,0)$ with P_1.

(b) Two collinear faces

Let us translate P_2 as described in the convex case upwards so that P_2 lies on the line joining P_1 to $(0,\nu)$. The Newton diagram still has three faces—$(0,\nu)$ to P_2, P_2 to P_1 and P_1 to $(\mu,0)$—but now those coincident with P_2 are collinear (see figure 4(a)). It still holds that δ_1 and δ_2 are positive, and Δ is positive. The significant difference from the convex case is that one of the two inequalities comprising (5.3) fails, and we have $\nu = \Delta/M$.

As both δ_1 and δ_2 are still positive, we proceed to displace integration contours in (5.1) as before, which yields the same two series of contributions I_1 and I_2 given in (5.6) and (5.7), respectively. Indeed, the analysis of the series I_2 carries over without alteration in the present circumstance, but an examination of the logarithm of the modulus of the integrands in (5.6) gives a large ρ behaviour for (5.8) of 0 (measured against the scale $\rho\log\rho$). Furthermore, the powers of λ in the integrands of (5.6) are now independent of t_1:

$$\delta_1 t_1 + \delta_2 t_2^{(1)} = \frac{\delta_2}{m_2}K$$

(see (5.12)).

Thus, I_1 can be written as

$$I_1 = \frac{1}{m_2\nu}\sum_k \frac{(-1)^k}{k!}J(k)\lambda^{-\delta_2 K/m_2},$$

with

$$J(k) \equiv \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t)\Gamma\left(\frac{K-m_1 t}{m_2}\right)\Gamma\left(\frac{m_2 - n_2 K + \Delta t}{m_2\nu}\right)\mathrm{d}t.$$

By employing (2.1), with $\theta = \pm\frac{1}{2}\pi$, it is easy to see that this integral converges and further, for $t = \rho\mathrm{e}^{\mathrm{i}\theta}$ with $|\theta| < \frac{1}{2}\pi$, the logarithm of the modulus of the integrand of $J(k)$ is seen exhibit the dominant large-ρ behaviour

$$\rho\cos\theta\log\left[\left(\frac{\Delta}{m_2\nu}\right)^{\Delta/(m_2\nu)}\left(\frac{m_2}{m_1}\right)^{m_1/m_2}\right].$$

The term inside the logarithm can be seen to be less than unity by using the fact that $\Delta = \nu M$ and the hypothesis $m_2 < m_1$, from which we conclude that a convergent series representation for $J(k)$ can be obtained by displacing the integration contour to the right. In doing this, we encounter poles at $(K + m_2 l)/m_1$ for non-negative integers l, and at $n_2K/\Delta - m_2L'/\Delta$, but only for those non-negative integers l rendering this last expression positive. Observe that the latter expression is $t_1^{(2)}$ of (5.9), with the complementary inequality used there.

After displacing the t contour for $J(k)$ to the right, we arrive at the evaluation

$$J(k) = \frac{m_2}{m_1}s_k - \frac{m_2\nu}{\Delta}\sum_{l'}\frac{(-1)^l}{l!}f_1(k,l), \tag{5.23}$$

where the prime on the index l in the finite sum indicates the restriction discussed in the previous paragraph, and where we have set

$$s_k = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\Gamma\left(\frac{K+m_2 l}{m_1}\right)\Gamma\left(\frac{m_1 - n_1 K + \Delta l}{m_1\nu}\right), \tag{5.24}$$

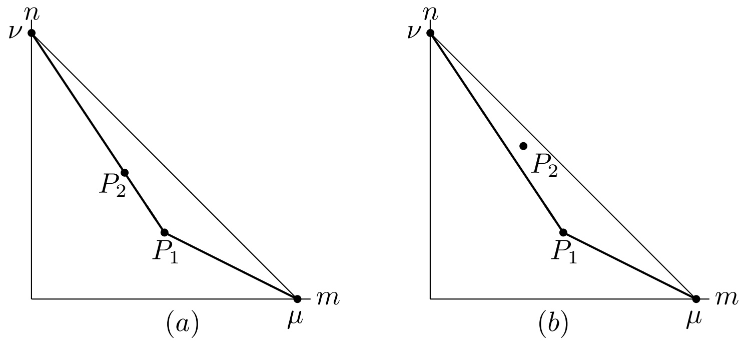

Figure 4. Newton diagrams for two internal points: (*a*) the collinear case and (*b*) P_2 behind the Newton diagram but in front of the back face.

$$f_1(k,l) = \Gamma\left(\frac{n_2K - m_2L'}{\Delta}\right)\Gamma\left(\frac{-n_1K + m_1L'}{\Delta}\right).$$

Use of the representation (5.23) together with the expansions (5.18) and (5.21) then gives the large-λ expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_1\nu}\sum_k \frac{(-1)^k}{k!}s_k\lambda^{-\delta_2K/m_2} - \frac{1}{\Delta}\sum_{k,l'}\frac{(-1)^{k+l}}{k!l!}f_1(k,l)\lambda^{-\delta_2K/m_2}\right.$$
$$\left. + \frac{1}{\Delta}\sum_{k,l'} f_{12}(k,l) + \frac{1}{\mu n_1}\sum_{k,l} f_{21}(k,l)\right\}.$$

Some further simplification is possible, after the fashion preceding (5.22). Observe that after relabelling $k \to l$, $l \to k$ in the sum for I_{22} (corresponding to the second sum above with restricted argument), we find that the two restricted sums, both with leading factors of $\pm 1/\Delta$, have exactly the same range of arguments. Furthermore, elementary arithmetic shows, for the case where P_2 lies on the line joining P_1 and $(0,\nu)$, that $-\delta_2/m_2 = (\Delta - \mu N)/(\mu\Delta)$. Thus, the two series involving leading factors of $\pm 1/\Delta$ annihilate, leaving the expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_1\nu}\sum_k \frac{(-1)^k}{k!}s_k\lambda^{-\delta_2K/m_2} + \frac{1}{\mu n_1}\sum_{k,l} f_{21}(k,l)\right\},$$

where f_{21} is given in (5.19) and s_k is defined in (5.24). The first series in this expansion can be associated with the face of the Newton diagram which joins $(0,\nu)$ to P_1 (and which contains the vertex P_2 as well), and the second with the face connecting the vertices P_1 and $(\mu, 0)$.

(*c*) *One internal point behind the Newton diagram*

Let us suppose that P_2 now lies behind the Newton diagram, but in front of the back face (see figure 4*b*). We suppose still that $\mu > m_1 > m_2$ and $n_1 < n_2 < \nu$ as before. In this circumstance, Δ remains positive, and both δ_1 and δ_2 are positive.

If we proceed as in the convex case, and select again t_2 as the integration variable to start with in (5.1), then we find that the t_2 contour is to be displaced to the right, with appropriate contributions to the asymptotics of $I(\lambda)$ stemming from the $t_2^{(1)}$ and $t_2^{(2)}$ series as before (cf. (5.4) and (5.5)). The resulting formal asymptotic series (5.6) and (5.6) must therefore still apply in the present case.

If we examine the logarithms of the moduli of the integrals appearing in I_1 and I_2, we find that, whereas the integrals for the I_2 series still have their integration contours displaced to the right, the integrals in the I_1 series must now have their contours displaced to the *right*, in marked contrast to the situation occuring in the convex case. To see this, note that the integrals in (5.6) have integrands with logarithms of the dominant real part governed by

$$\rho\cos\theta\log\rho(\Delta - M\nu)/m_2\nu + \mathcal{O}(\rho),$$

which tends to ∞ as $\rho\to\infty$, in view of the fact that since the line through P_1 and P_2 now meets the n-axis above $(0,\nu)$, the inequality $\Delta > \nu M$ now holds, in contrast with the convex case (5.3).

Candidate poles for evaluating integrals in I_1 are to be found in the two sequences now: one from poles of $\Gamma(t_2^{(1)})$ (which was not used in the convex case), and one from poles of $\Gamma((1-n_1t_1-n_2t_2^{(1)})/\nu) = \Gamma((m_2-n_2K+\Delta t_1)/m_2\nu)$ (which was used in the convex case). The first sequence of poles is the $t_1^{(1)}$ sequence, with

$$t_1^{(1)} = (K+m_2l)/m_1; \tag{5.25}$$

the $t_1^{(2)}$ sequence is given by (5.9), with the restriction $t_1^{(2)} > 0$ holding, in view of the displacement of the integration contour to the right.

Poles from (5.25) give rise to the formal asymptotic series

$$\begin{aligned} I_{11} &= \frac{1}{m_1\nu}\sum_{k,l}\frac{(-1)^{k+l}}{k!l!}\Gamma(t_1^{(1)})\Gamma\left(\frac{1-n_1t_1^{(1)}-n_2t_2^{(1)}(t_1^{(1)})}{\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}} \\ &= \frac{1}{m_1\nu}\sum_{k,l}f_{11}^*(k,l), \end{aligned}$$

where

$$f_{11}^*(k,l) \equiv \frac{(-1)^{k+l}}{k!l!}\Gamma\left(\frac{K+m_2l}{m_1}\right)\Gamma\left(\frac{m_1-n_1K+\Delta l}{m_1\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \tag{5.26}$$

with

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\delta_1t_1^{(1)} - \delta_2t_2^{(1)}(t_1^{(1)}) = -\frac{\delta_1}{m_1}K + \frac{\nu M-\Delta}{m_1\nu}l. \tag{5.27}$$

Poles from (5.9) give rise to $-I_{12}$, with I_{12} as in (5.13), (5.14) and (5.15), but with the constraint replaced by $t_1^{(2)} > 0$.

The $t_2^{(2)}$ sequence, of course, offers up the contributions I_{21} and I_{22} as before (in (5.18), (5.21) and associated equations). The restriction on k and l accompanying I_{22} is the same as that for $-I_{12}$, so rather than adding together, as was the case in the convex setting, these two series annihilate, leaving us with the large-λ expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}(I_{11}+I_{21}),$$

or

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_1\nu}\sum_{k,l}f_{11}^*(k,l) + \frac{1}{\mu n_1}\sum_{k,l}f_{21}(k,l)\right\}, \tag{5.28}$$

with f_{11}^* and f_{21} given by (5.11), (5.12), (5.26) and (5.27). The correspondence with the sides of the Newton polygon is again readily apparent: the first series in (5.28) is associated with the face defined by $(0,\nu)$ and P_1, and the second with the face defined by $(\mu,0)$ and P_1.

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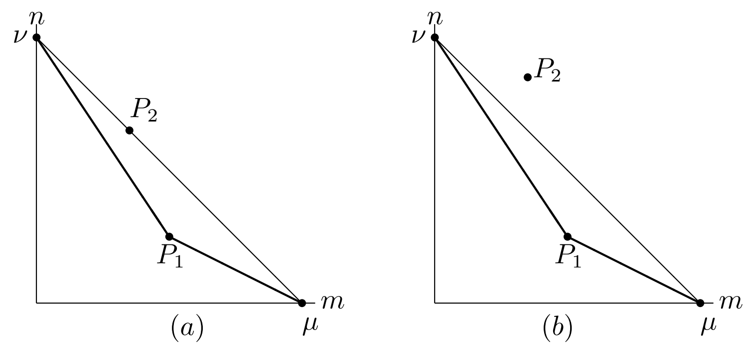

Figure 5. Newton diagrams for two internal points: (*a*) P_2 on the back face; and (*b*) P_2 behind the back face.

(*d*) *One internal point on or behind the back face*

We suppose in this section that P_2 either lies on the back face joining $(\mu, 0)$ to $(0, \nu)$ (as depicted in figure 5*a*), or behind it (as in figure 5*b*); in this setting, it follows that $\delta_2 = 0$ or $\delta_2 < 0$, respectively.

If P_2 is on the back face, then δ_2 vanishes, and the integrand (5.1) has no t_2-dependence. Accordingly, if we elect to displace the t_2 contour first, then we shall have to examine the lower order terms in the dominant part of the logarithm of the modulus of the integrand, given in (3.6), in order to determine the direction in which the integration contour is to be translated. This additional complication can be avoided by considering the t_1 integral first. For the t_1 integral, with $t_1 = \rho e^{i\theta}$, $|\theta| < \frac{1}{2}\pi$, we find that the logarithm of the modulus of the integrand behaves as

$$\delta_1 \rho \cos\theta \log \rho + \mathcal{O}(\rho),$$

for large ρ. Since P_1 lies in front of the back face, $\delta_1 > 0$, and so we see that the above estimate tends to ∞ as $\rho \to \infty$. Accordingly, we displace the t_1 contour to the right. Candidate poles arise in two sequences: from poles of $\Gamma((1 - m_1 t_1 - m_2 t_2)/\mu)$ we have

$$t_1^{(1)} = (K - m_2 t_2)/m_1,$$

and from poles of $\Gamma((1 - n_1 t_1 - n_2 t_2)/\nu)$ we have

$$t_1^{(2)} = (K' - n_2 t_2)/n_1; \tag{5.29}$$

as usual, k is a non-negative integer, and the sequences of poles we generate throughout are assumed to be distinct.

The $t_1^{(1)}$ sequence of poles gives rise to the formal series (suppressing a leading factor of $\lambda^{-1/\mu - 1/\nu}$)

$$I_1 = \frac{1}{m_1 \nu} \sum_k \frac{(-1)^k}{k!} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(t_1^{(1)}) \Gamma(t_2) \Gamma\left(\frac{1 - n_1 t_1^{(1)} - n_2 t_2}{\nu}\right) \lambda^{-\delta_1 t_1^{(1)}}\, dt_2, \tag{5.30}$$

and similarly, $t_1^{(2)}$ gives rise to

$$I_2 = \frac{1}{\mu n_1} \sum_k \frac{(-1)^k}{k!} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(t_1^{(2)}) \Gamma(t_2) \Gamma\left(\frac{1 - m_1 t_1^{(1)} - m_2 t_2}{\mu}\right) \lambda^{-\delta_1 t_1^{(2)}}\, dt_2. \tag{5.31}$$

The integrals appearing in the I_1 series (5.30) have integrands which reduce to

$$\Gamma(t_2)\Gamma\left(\frac{K-m_2t_2}{m_1}\right)\Gamma\left(\frac{m_1-n_1K-\Delta t_2}{m_1\nu}\right)\lambda^{-\delta_1 t_1^{(1)}},$$

which, with $t_2=\rho e^{i\theta}$, $|\theta|<\frac{1}{2}\pi$, has logarithm of the modulus governed by

$$\rho\cos\theta\log\rho(M\nu-\Delta)/m_1\nu+\mathcal{O}(\rho),$$

for large ρ. Because $\Delta/M>\nu$ in the case $\delta_2\leqslant 0$, this estimate tends to $-\infty$ as $\rho\to\infty$. Therefore, the t_2 contour must be displaced to the left to obtain asymptotic behaviour. We find that there are two sequences of t_2 poles that contribute: $t_2^{(1)}=-l$, and $t_2^{(2)}=(-n_1K+m_1L')/\Delta$; here, l is a non-negative integer, and $t_2^{(2)}$ must be subject to the restriction $t_2^{(2)}\leqslant 0$.

The $t_2^{(1)}$ sequence of poles gives rise to the formal asymptotic series

$$I_{11}=\frac{1}{m_1\nu}\sum_{k,l}f_{11}^*(k,l), \tag{5.32}$$

where f_{11}^* is given in (5.26), and $-\boldsymbol{\delta}\cdot\boldsymbol{t}=-\delta_1t_1^{(1)}(t_2^{(1)})=-\delta_1(K+m_2l)/m_1$ is also given by (5.27); to see this latter point, apply the identities

$$m_2\delta_1-m_1\delta_2=\Delta/\nu-M\quad\text{and}\quad n_1\delta_2-n_2\delta_1=\Delta/\mu-N. \tag{5.33}$$

The $t_2^{(2)}$ sequence yields the formal asymptotic sum

$$I_{12}=-\frac{1}{\Delta}\sum_{k,l'}f_{12}(k,l), \tag{5.34}$$

where f_{12} is given in (5.14) and $\boldsymbol{\delta}\cdot\boldsymbol{t}$ is given in (5.15). The prime on l indicates the presence of the restriction on the range of l dictated by $t_2^{(2)}\leqslant 0$.

The integrals appearing in the I_2 series (5.31) have integrands which reduce to

$$\Gamma(t_2)\Gamma\left(\frac{K'-n_2t_2}{n_1}\right)\Gamma\left(\frac{n_1-m_1K'+\Delta t_2}{\mu n_1}\right)\lambda^{-\delta_1 t_1^{(2)}},$$

which, with $t_2=\rho e^{i\theta}$, $|\theta|<\frac{1}{2}\pi$, has logarithm of the modulus governed by

$$\rho\cos\theta\log\rho(\Delta-\mu N)/\mu n_1+\mathcal{O}(\rho),$$

for large ρ. Unlike the situation for I_1, the first of the inequalities in (5.3) continues to hold, from which we see that the above estimate tends to $-\infty$ as $\rho\to\infty$ and so the t_2 contour must be displaced to the left. Candidate poles arise again in two sequences, $t_2^{(1)}=-l$ and $t_2^{(2)}=(-n_1K+m_1L')/\Delta$; as before, l is a non-negative integer and $t_2^{(2)}$ is subject to the constraint $t_2^{(2)}\leqslant 0$.

The $t_2^{(1)}$ sequence of poles gives rise to the formal asymptotic series

$$I_{21}=\frac{1}{m_1\nu}\sum_{k,l}f_{21}(k,l), \tag{5.35}$$

where f_{21} is given in (5.19) and $\boldsymbol{\delta}\cdot\boldsymbol{t}$ given in (5.20). (Use (5.33) in $-\boldsymbol{\delta}\cdot\boldsymbol{t}=-\delta_1t_1^{(2)}(t_2^{(1)})=-\delta_1(K'+n_2l)/n_1$ to see this.) The $t_2^{(2)}$ sequence yields the formal asymptotic sum

$$I_{22}=\frac{1}{\Delta}\sum_{k,l'}f_{12}(l,k), \tag{5.36}$$

where f_{12} is given in (5.14), and $\boldsymbol{\delta}\cdot\boldsymbol{t}$ is given in (5.15).

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Collecting expansions (5.32), (5.34), (5.35) and (5.36), and restoring the leading factor of $\lambda^{-1/\mu-1/\nu}$, we obtain the asymptotic expansion

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}(I_{11}+I_{12}+I_{21}+I_{22}),$$

for $\lambda \to \infty$. Further reduction is possible: observe that the inequalities governing the restrictions on the sums in I_{12} and I_{22} are identical (after relabelling the indices in one of the sums), and that I_{12} is the negative of I_{22}. Therefore, these two series annihilate to yield finally

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_1\nu}\sum_{k,l} f^*_{11}(k,l)+\frac{1}{\mu n_1}\sum_{k,l} f_{21}(k,l)\right\}, \tag{5.37}$$

as $\lambda \to \infty$. For the case of $\delta_2 < 0$, we find that the same expansion (5.37) results and we omit the derivation.

We mention that the expansions obtained in the cases where P_2 was behind the Newton diagram (either in front, on, or behind the back face) can also all be developed through termwise integration. Beginning with

$$I(\lambda)=\int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu+x^{m_1}y^{n_1}+y^\nu)]\exp(-\lambda x^{m_2}y^{n_2})\,\mathrm{d}x\mathrm{d}y,$$

we develop the exponential factor involving the monomial $x^{m_2}y^{n_2}$ into its Maclaurin series and interchange the order of integration and summation to arrive at

$$I(\lambda) \sim \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\lambda^k I_k(\lambda), \tag{5.38}$$

where

$$\begin{aligned} I_k(\lambda) &= \int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu+x^{m_1}y^{n_1}+y^\nu)]x^{km_2}y^{kn_2}\,\mathrm{d}x\mathrm{d}y \\ &= \frac{\lambda^{-(1+km_2)/\mu-(1+kn_2)/\nu}}{\mu\nu}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t)\Gamma\left(\frac{1+km_2-m_1t}{\mu}\right) \\ &\qquad\times\Gamma\left(\frac{1+kn_2-n_1t}{\nu}\right)\lambda^{-\delta_1 t}\,\mathrm{d}t. \end{aligned}$$

Since P_1 is in front of the back face, the contour for I_k must be displaced to the right. Proceeding as we have done for the case of a single internal point (cf. § 4 *a*), we obtain an asymptotic expansion for each I_k, which must then be inserted into (5.38) to produce the previously obtained asymptotic expansions for $I(\lambda)$.

All the expansions obtained in §§ 5 *a*–*d* have repeatedly made use of the assumption that sequences of poles encountered in each case under analysis were simple. In most applications, however, double poles can be expected to arise, and are a certainty whenever μ and ν are rational. In these circumstances, one proceeds as we have done here with appropriate steps taken to compute residues of double poles. The presence of double poles in residue calculations will generate $\log\lambda$ terms in the asymptotic expansions produced. Examples illustrating how one deals with violations of the simple-poles-only hypothesis are presented in § 7.

(*e*) *Remoteness and order of leading terms*

As we have seen, when one of the internal points lies behind the Newton diagram, only two expansions result (one for each face of the diagram), and upon closer

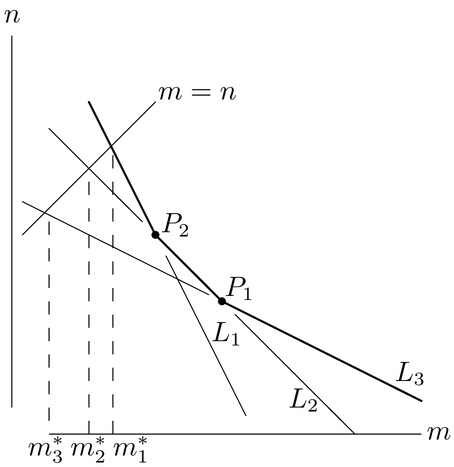

Figure 6. Portion of the Newton diagram for two internal points in the case where P_1 and P_2 both lie below the diagonal $m=n$.

examination of the leading terms for each such expansion, it is apparent that the coordinates of the point behind the Newton diagram do not appear (set $k=l=0$ in the expansions to see this). Thus, the treatment of remoteness in this case is identical to that for the case of one internal point in front of the back face (cf. § 4 *c*).

Consequently, we need only examine remoteness in the convex case detailed in § 5 *a*. For the convex case, the leading terms in the asymptotic expansion of $I(\lambda)$ are given by (cf. equations (5.22), (5.11), (5.14) and (5.19))

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_2\nu}\Gamma\left(\frac{1}{m_2}\right)\Gamma\left(\frac{m_2-n_2}{m_2\nu}\right)\lambda^{-\delta_2/m_2}+\frac{1}{\Delta}\Gamma\left(\frac{n_2-m_2}{\Delta}\right)\right.$$
$$\left.\times\Gamma\left(\frac{m_1-n_1}{\Delta}\right)\lambda^{(\Delta-\mu N)/\mu\Delta+(\Delta-M\nu)/\nu\Delta}+\frac{1}{\mu n_1}\Gamma\left(\frac{1}{n_1}\right)\Gamma\left(\frac{n_1-m_1}{\mu n_1}\right)\lambda^{-\delta_1/n_1}\right\}. \tag{5.39}$$

There are only three possible cases to address: both internal points P_1 and P_2 lie below the $m=n$ line; the points 'straddle' the $m=n$ line (i.e. one lies above the line and one lies below); and both lie above the $m=n$ line. By symmetry considerations, we can reduce this further by discarding one of the cases where both points lie on the same side of the diagonal. Let m_1^*, m_2^* and m_3^* be the abscissas of the points of intersection of the line $m=n$ with each of the lines L_1, L_2 and L_3, defined, respectively, as the line generated by $(0,\nu)$ and P_2, P_2 and P_1, and finally P_1 and $(\mu,0)$. Elementary analytical geometry gives us

$$m_1^*=\frac{m_2\nu}{m_2+\nu-n_2},\quad m_2^*=\frac{\Delta}{M+N},\quad m_3^*=\frac{\mu n_1}{\mu-m_1+n_1}.$$

After distributing the factor $\lambda^{-1/\mu-1/\nu}$ across all three terms in (5.39), we see that the orders of the leading terms in (5.39) have a particularly elegant expression in terms of the quantities m_1^*, m_2^*, m_3^*: the first term has order $-1/m_1^*$, the second has order $-1/m_2^*$ and the third has order $-1/m_3^*$.

Let us suppose, then, that both points P_1 and P_2 lie below the $m=n$ line (see figure 6). Evidently, we have the ordering $m_3^*<m_2^*<m_1^*$, whence $-1/m_3^*<-1/m_2^*<-1/m_1^*$. The dominant leading term is therefore the one of order $-1/m_1^*$, correspond-

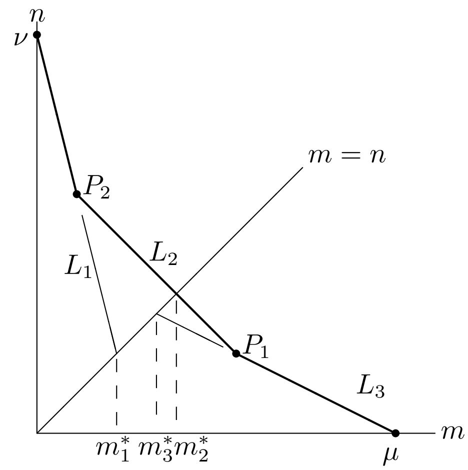

Figure 7. The Newton diagram for two internal points in the case where P_1 and P_2 'straddle' the diagonal $m = n$.

ing to the remoteness of the Newton diagram. For this case, $I(\lambda)$ has the dominant asymptotic behaviour

$$I(\lambda) \sim \frac{\lambda^{-1/m_1^*}}{m_2 \nu} \Gamma\left(\frac{1}{m_2}\right) \Gamma\left(\frac{m_2 - n_2}{m_2 \nu}\right),$$

for $\lambda \to \infty$. Observe that all arguments of the Γ functions in this leading term are positive in view of the fact that P_2 below the $m = n$ line implies $m_2 > n_2$.

When the points P_1 and P_2 straddle the diagonal $m = n$, as depicted in figure 7, we arrive at a different ordering of the abscissas m_i^*. In this case, we find $m_1^* < m_3^* < m_2^*$ which in turn yields $-1/m_1^* < -1/m_3^* < -1/m_2^*$. From this, it is apparent that the dominant leading term in (5.39) is the second one corresponding to $-1/m_2^*$, namely

$$I(\lambda) \sim \frac{\lambda^{-1/m_2^*}}{\Delta} \Gamma\left(\frac{n_2 - m_2}{\Delta}\right) \Gamma\left(\frac{m_1 - n_1}{\Delta}\right).$$

The quantity $-1/m_2^*$ is the remoteness of the Newton diagram in this instance. Since P_2 lies above the diagonal and P_1 lies below the diagonal, we must also have $n_2 > m_2$ and $m_1 > n_1$ so that the Γ functions in the leading term have positive arguments.

(*f*) *Geometric interpretation of the asymptotic scales*

The various asymptotic scales of λ (displayed as (5.12), (5.15) and (5.20)) found in the asymptotic sums over k and l in (5.22), can be given a simple geometric interpretation based on the Newton diagram. From the result for the perpendicular distance p from a point (x', y') to a given line $Ax + By = C$ given by

$$p = \frac{C - Ax' - By'}{\sqrt{A^2 + B^2}}, \quad (C > 0),$$

we easily see that $\delta_i = p_i/p_0$ $(i = 1, 2)$, where

$$p_i = \frac{1 - m_i/\mu - n_i/\nu}{\sqrt{1/\mu^2 + 1/\nu^2}}, \quad p_0 = \frac{1}{\sqrt{1/\mu^2 + 1/\nu^2}}$$

are the perpendicular distances of the vertex $P_i = (m_i, n_i)$ and the origin, respectively, to the back face presented in figure 8 as $\overline{P_0 P_3}$, with $P_0 \equiv (\mu, 0)$ and $P_3 \equiv (0, \nu)$.

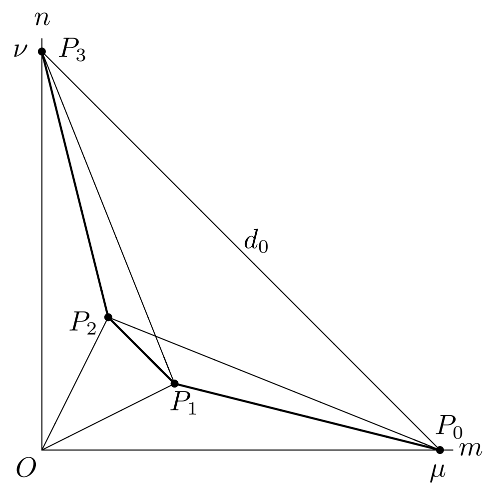

Figure 8. Triangular regions in the Newton diagram for the convex case with two internal points.

If we write $p_0 = \mu\nu/d_0$, where $d_0 = P_0P_3$, the length of the back face, then we find the powers associated with two of these asymptotic scales given by

$$\frac{\delta_1\nu}{n_1} = \frac{p_1 d_0}{\mu n_1} = \frac{\text{area}\,\triangle P_0P_3P_1}{\text{area}\,\triangle OP_0P_1}$$

and

$$\frac{\delta_2\mu}{m_2} = \frac{p_2 d_0}{m_2\nu} = \frac{\text{area}\,\triangle P_0P_3P_2}{\text{area}\,\triangle OP_2P_3}.$$

To interpret the powers of the remaining scales, we note that the perpendicular distance from P_1 to the line segment $\overline{P_0P_2}$ is p_{11}, where

$$p_{11} = \frac{n_1 + n_2m_1/(\mu - m_2) - \mu n_2/(\mu - m_2)}{\sqrt{1 + n_2^2/(\mu - m_2)^2}} = \frac{\Delta - \mu N}{d_1},$$

with $d_1 = P_0P_2$, the length of $\overline{P_0P_2}$. Then

$$\frac{\Delta - \mu N}{\mu n_1} = \frac{p_{11}d_1}{\mu n_1} = \frac{\text{area}\,\triangle P_0P_2P_1}{\text{area}\,\triangle OP_0P_1},$$

and, in a similar fashion,

$$\frac{\Delta - \nu M}{m_2\nu} = \frac{p_{22}d_2}{m_2\nu} = \frac{\text{area}\,\triangle P_3P_1P_2}{\text{area}\,\triangle OP_2P_3},$$

where p_{22} denotes the perpendicular distance from P_2 to the line segment $\overline{P_1P_3}$ and $d_2 = P_1P_3$. Finally, since the area of triangle OP_1P_2 is $\frac{1}{2}\Delta$, we have

$$\frac{\Delta - \mu N}{\Delta} = \frac{p_{11}d_1}{\Delta} = \frac{\text{area}\,\triangle P_0P_1P_2}{\text{area}\,\triangle OP_1P_2}$$

and

$$\frac{\Delta - \nu M}{\Delta} = \frac{\text{area}\,\triangle P_3P_1P_2}{\text{area}\,\triangle OP_1P_2}.$$

Hence the powers associated with the different asymptotic scales of λ appearing in the expansion (5.22) can be interpreted as ratios of triangular areas in the Newton diagram.

6. Asymptotics with three or more internal points

In view of the fact that internal points do not give rise to additional series constituting the asymptotic expansion of $I(\lambda)$, we shall restrict our attention in this section to the purely convex case, i.e. the case where all internal points lie on the Newton diagram, and no two faces are collinear. We will find from our analysis of the case of three internal points a high degree of structure in the form of the resulting expansions. This in turn will enable us to record an algorithm for generating all terms in the expansion of $I(\lambda)$ for any number of internal points in the convex case. As was the case in earlier sections, we will assume that all poles of integrands are simple.

(*a*) *Three internal points in the convex case*

Now let us suppose $k = 3$ in (3.1), and that the internal points $P_i = (m_i, n_i)$, $i = 1, 2, 3$, are labelled so that $\mu > m_1 > m_2 > m_3$ and $n_1 < n_2 < n_3 < \nu$. The points P_1, P_2 and P_3 then appear as shown in figure 9. We shall find it convenient to define analogues of M, N and Δ from the previous section, and supply a small set of identities relating these quantities. For the quantities

$$M_{ij} = m_i - m_j, \quad N_{ij} = n_j - n_i, \quad \Delta_{ij} = m_i n_j - n_i m_j, \tag{6.1}$$

where $i, j = 1, 2, 3$, $i < j$, it is easily seen that

$$\left.\begin{aligned} m_i\delta_j - m_j\delta_i = M_{ij} - \Delta_{ij}/\nu, \quad n_j\delta_i - n_i\delta_j = N_{ij} - \Delta_{ij}/\mu, \\ m_3\Delta_{12} + m_1\Delta_{23} = m_2\Delta_{13}, \quad n_3\Delta_{12} + n_1\Delta_{23} = n_2\Delta_{13} \end{aligned}\right\} \tag{6.2}$$

and

$$\delta_1\Delta_{23} - \delta_2\Delta_{12} + \delta_3\Delta_{12} = \Delta_{12} - \Delta_{13} + \Delta_{23}.$$

With the ordering imposed on the points P_1, P_2 and P_3, we have all the M_{ij}, N_{ij} positive, and the quantities Δ_{ij} can be seen to be the areas of parallelograms generated by pairs of position vectors $\boldsymbol{P}_i$, $\boldsymbol{P}_j$ which form positively ordered bases for the mn-plane. Accordingly, each Δ_{ij} is positive.

The lines constituting the faces in the Newton diagram meet the coordinate axes in a regular fashion, giving rise to a number of useful inequalities. The intersection points on the m-axis, indicated in figure 9 by labels $(1_m), (2_m), \ldots$, yields the chain of inequalities

$$\frac{m_3\nu}{\nu - n_3} < \frac{\Delta_{23}}{N_{23}} < \frac{\Delta_{13}}{N_{13}} < \frac{\Delta_{12}}{N_{12}} < \mu,$$

while those intersection points on the n-axis, indicated in the figure by labels $(1_n), (2_n), \ldots$, give

$$\frac{\mu n_1}{\mu - m_1} < \frac{\Delta_{12}}{M_{12}} < \frac{\Delta_{13}}{M_{13}} < \frac{\Delta_{23}}{M_{23}} < \nu. \tag{6.3}$$

Under the simplifying assumption $c_1 = c_2 = c_3 = 1$, (3.1) for $k = 3$ becomes

$$I(\lambda) = \frac{\lambda^{-1/\mu - 1/\nu}}{\mu\nu}\left(\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\right)^3 \Gamma(t_1)\Gamma(t_2)\Gamma(t_3)\Gamma\left(\frac{1 - \boldsymbol{m}\cdot\boldsymbol{t}}{\mu}\right) \times\Gamma\left(\frac{1 - \boldsymbol{n}\cdot\boldsymbol{t}}{\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3, \tag{6.4}$$

where the vector representations in (3.3) are being used. If we begin our investigation

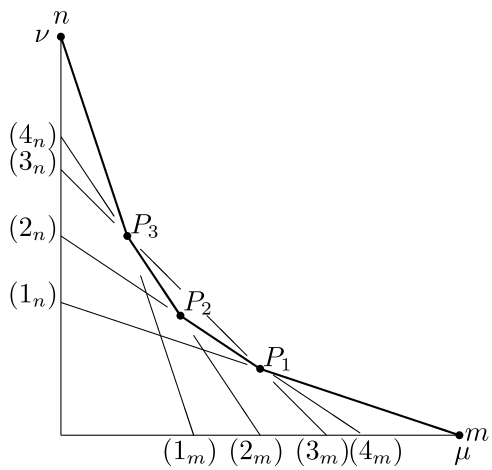

Figure 9. The Newton diagram for three internal points in the convex case. The values of the labels $(1_m), (2_m), \ldots$, are discussed in the text.

by displacing the t_3 contour first, setting $t_3 = \rho e^{i\theta}$ with $|\theta| < \frac{1}{2}\pi$, then we find that the logarithm of the modulus of the integrand is dominated by

$$\delta_3 \rho \cos\theta \log\rho + \mathcal{O}(\rho),$$

for large ρ. Since all the points P_i in the Newton diagram lie in front of the back face, the quantity δ_3 is positive, from which it follows the above estimate must tend to $+\infty$ as $\rho \to \infty$. The integration contour in turn must be displaced to the right to develop asymptotic expansions.

Candidate poles arise in two sequences (assumed to share no points):

$$t_3^{(1)} = (K - m_1 t_1 - m_2 t_2)/m_3, \quad k = 0, 1, 2, \ldots, \quad \text{and}$$
$$t_3^{(2)} = (K' - n_1 t_1 - n_2 t_2)/n_3, \quad k = 0, 1, 2, \ldots.$$

Poles from the $t_3^{(1)}$ sequence give rise to the series of contributions (suppressing, as before, the leading factor of $\lambda^{-1/\mu - 1/\nu}$)

$$I_1 = \frac{1}{m_3 \nu} \sum_k \frac{(-1)^k}{k!} \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \right)^2 \Gamma(t_1)\Gamma(t_2)\Gamma(t_3^{(1)}) \times \Gamma\left(\frac{1 - n_1 t_1 - n_2 t_2 - n_3 t_3^{(1)}}{\nu} \right) \lambda^{-\boldsymbol{\delta} \cdot \boldsymbol{t}} \, dt_1 dt_2, \tag{6.5}$$

with $\boldsymbol{\delta} \cdot \boldsymbol{t} = \delta_1 t_1 + \delta_2 t_2 + \delta_3 t_3^{(1)}$. Similarly, the $t_3^{(2)}$ sequence of poles yields the series

$$I_2 = \frac{1}{\mu n_3} \sum_k \frac{(-1)^k}{k!} \left(\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \right)^2 \Gamma(t_1)\Gamma(t_2)\Gamma(t_3^{(2)}) \times \Gamma\left(\frac{1 - m_1 t_1 - m_2 t_2 - m_3 t_3^{(2)}}{\mu} \right) \lambda^{-\boldsymbol{\delta} \cdot \boldsymbol{t}} \, dt_1 dt_2, \tag{6.6}$$

with $\boldsymbol{\delta} \cdot \boldsymbol{t} = \delta_1 t_1 + \delta_2 t_2 + \delta_3 t_3^{(2)}$.

The integrals in the I_1 series (6.5) have integrands reducing to

$$\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{K-m_1t_1-m_2t_2}{m_3}\right)\Gamma\left(\frac{m_3-n_3K+\Delta_{13}t_1+\Delta_{23}t_2}{m_3\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

from which it follows, upon setting $t_2=\rho\mathrm{e}^{\mathrm{i}\theta}$ with $|\theta|<\frac{1}{2}\pi$, the logarithm of the modulus of the integrand behaves as

$$\rho\cos\theta\log\rho(\Delta_{23}-M_{23}\nu)/m_3\nu+\mathcal{O}(\rho),$$

for large ρ. From (6.3), we have $\Delta_{23}<M_{23}\nu$ whence it follows that the above estimate tends to $-\infty$ as $\rho\to\infty$. The next step in determining the asymptotics arising from (6.5) involves displacing the t_2 contour to the left. Because of the presence of an additional Γ function in the integrand, there are now three sequences of poles that must be considered: poles of $\Gamma(t_2)$, poles of $\Gamma((K-m_1t_1-m_2t_2)/m_3)$ and poles of $\Gamma((m_3-n_3K+\Delta_{12}t_1+\Delta_{23}t_2)/m_3\nu)$. However, poles of the second Γ function listed, of the form $(K+m_3l-m_1t_1)/m_2$, l a non-negative integer, fail to satisfy the requirement that the real part be negative (recall that along the integration contour in the t_1 plane, $\operatorname{Re}t_1=0$, except for an indentation to the right near the origin where $\operatorname{Re}t_1$ is positive, but arbitrarily small). As a result there are only two sequences of poles that can contribute to the asymptotic series arising from (6.5):

$$t_2^{(1)}=-l,$$
$$t_2^{(2)}=-\frac{m_3}{\Delta_{23}}L'+\frac{n_3}{\Delta_{23}}K-\frac{\Delta_{13}}{\Delta_{23}}t_1\leqslant 0, \tag{6.7}$$

where l is a non-negative integer, and the poles listed in (6.7) are subject to the indicated restrictions. These sequences of poles give rise to the formal series (with suppressed factor of $\lambda^{-1/\mu-1/\nu}$)

$$I_{11}=\frac{1}{m_3\nu}\sum_{k,l}\frac{(-1)^{k+l}}{k!l!}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t_1)\Gamma(t_3^{(1)}(-l))$$
$$\times\Gamma\left(\frac{1-n_1t_1+n_2l-n_3t_3^{(1)}(-l)}{\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \tag{6.8}$$

$$I_{12}=\frac{1}{\Delta_{23}}\sum_{k,l}\frac{(-1)^{k+l}}{k!l!}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t_1)\Gamma(t_2^{(3)})\Gamma(t_3^{(1)}(t_2^{(3)}))\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}\,\mathrm{d}t_1, \tag{6.9}$$

with integrands that reduce to

$$\Gamma(t_1)\Gamma\left(\frac{K-m_1t_1+m_2l}{m_3}\right)\Gamma\left(\frac{m_3-n_3K-\Delta_{23}l+\Delta_{13}t_1}{m_3\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

$$\Gamma(t_1)\Gamma\left(\frac{n_3K-m_3L'-\Delta_{13}t_1}{\Delta_{23}}\right)\Gamma\left(\frac{-n_2K+m_2L'+\Delta_{12}t_1}{\Delta_{23}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

and have logarithms of moduli of integrands with the large-ρ behaviour (setting, in each case, $t_1=\rho\mathrm{e}^{\mathrm{i}\theta}$, $|\theta|<\frac{1}{2}\pi$)

$$\rho\cos\theta\log\rho(\Delta_{13}-M_{13}\nu)/m_3\nu+\mathcal{O}(\rho),$$
$$\rho\cos\theta\log\rho(\Delta_{23}-\Delta_{13}+\Delta_{12})/\Delta_{23}+\mathcal{O}(\rho),$$

respectively. Use of the inequalities (6.3) allows us to determine straightaway that the t_1 contour in (6.8) must be displaced to the left. For the series (6.9), however, we must determine the sign of the quantity $\Delta_{23} - \Delta_{13} + \Delta_{12}$ appearing in the last order estimate.

The quantity $\Delta_{23} - \Delta_{13} + \Delta_{12}$ is, in fact, a measure of relative convexity. To see this, consider the line through P_1 and P_3,

$$\frac{m}{M_{13}} + \frac{n}{N_{13}} - \frac{m_1}{M_{13}} - \frac{n_1}{N_{13}} = 0. \tag{6.10}$$

If P_2 lies below this line (i.e. on the same side of the line as the origin), then the left-hand side of (6.10) must be negative, whereas if P_2 lies above the line, the left-hand side must be positive. (If P_2 lies on (6.10), the Newton diagram has a pair of collinear faces, a case we have excluded from discussion.) Consequently, the case of three internal points in the convex case requires that

$$\frac{m_2}{M_{13}} + \frac{n_2}{N_{13}} - \frac{m_1}{M_{13}} - \frac{n_1}{N_{13}} = \frac{-\Delta_{13} + \Delta_{23} + \Delta_{12}}{M_{13}N_{13}} < 0,$$

from which we conclude that the t_1 contour in (6.9) must be translated to the left as is the case for (6.8).

For (6.8), we find there are two sequences of poles to consider,

$$\begin{aligned} t_1^{(1)} &= -r, \\ t_1^{(2)} &= \frac{n_3}{\Delta_{13}}K + \frac{\Delta_{23}}{\Delta_{13}}l - \frac{m_3}{\Delta_{13}}(1+\nu r) = \frac{n_3K + \Delta_{23}l - m_3R'}{\Delta_{13}} \leqslant 0, \end{aligned} \tag{6.11}$$

giving rise to the expansions (with the usual conventions)

$$I_{111} = \frac{1}{m_3\nu}\sum_{k,l,r} f_{111}(k,l,r), \quad I_{112} = \frac{1}{\Delta_{13}}\sum_{k,l,r'} f_{112}(k,l,r), \tag{6.12}$$

where

$$\begin{aligned} f_{111}(k,l,r) &= \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{K+m_1r+m_2l}{m_3}\right) \\ &\qquad \times\Gamma\left(\frac{m_3 - n_3K - \Delta_{13}r - \Delta_{23}l}{m_3\nu}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \\ -\boldsymbol{\delta}\cdot\boldsymbol{t} &= -\frac{\delta_3}{m_3}K + \frac{\Delta_{23} - M_{23}\nu}{m_3\nu}l + \frac{\Delta_{13} - M_{13}\nu}{m_3\nu}r, \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} f_{112}(k,l,r) &= \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{n_3K + \Delta_{23}l - m_3R'}{\Delta_{13}}\right) \\ &\qquad \times\Gamma\left(\frac{-n_1K + \Delta_{12}l + m_1R'}{\Delta_{13}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \\ -\boldsymbol{\delta}\cdot\boldsymbol{t} &= \frac{\Delta_{13} - \mu N_{13}}{\mu\Delta_{13}}K + \frac{\Delta_{13} - \Delta_{12} - \Delta_{23}}{\Delta_{13}}l + \frac{\Delta_{13} - M_{13}\nu}{\nu\Delta_{13}}R'. \end{aligned} \tag{6.14}$$

In these expressions, we have non-negative integer parameters k, l and r, and restrictions apply in those sums with primes on the summation indices. We have also set $R' \equiv 1 + \nu r$ (see (2.2)).

For (6.9), we find there are three sequences of poles to consider,

$$t_1^{(1)} = -r,$$

$$t_1^{(2)} = \frac{n_3}{\Delta_{13}}K - \frac{m_3}{\Delta_{13}}L' + \frac{\Delta_{23}}{\Delta_{13}}r \leqslant 0, \tag{6.15}$$

$$t_1^{(3)} = \frac{n_2}{\Delta_{12}}K - \frac{m_2}{\Delta_{12}}L' - \frac{\Delta_{23}}{\Delta_{12}}r \leqslant 0, \tag{6.16}$$

giving rise to the expansions (with the usual conventions regarding primed summation indices)

$$\begin{aligned} I_{121} &= \frac{1}{\Delta_{23}} \sum_{k,l',r} f_{121}(k,l,r), \\ I_{122} &= -\frac{1}{\Delta_{13}} \sum_{k,l',r'} f_{112}(k,r,l), \\ I_{123} &= \frac{1}{\Delta_{12}} \sum_{k,l',r'} f_{123}(k,l,r), \end{aligned} \tag{6.17}$$

where

$$\begin{aligned} f_{121}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} \Gamma\left(\frac{n_3K - m_3L' + \Delta_{13}r}{\Delta_{23}}\right) \\ \times \Gamma\left(\frac{-n_2K + m_2L' - \Delta_{12}r}{\Delta_{23}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \\ -\boldsymbol{\delta}\cdot\boldsymbol{t} = \frac{\Delta_{23} - \mu N_{23}}{\mu\Delta_{23}}K + \frac{\Delta_{23} - \nu M_{23}}{\nu\Delta_{23}}L' + \frac{\Delta_{12} - \Delta_{13} + \Delta_{23}}{\Delta_{23}}r, \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} f_{123}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!} \Gamma\left(\frac{n_2K - m_2L' - \Delta_{23}r}{\Delta_{12}}\right) \\ \times \Gamma\left(\frac{-n_1K + m_1L' + \Delta_{13}r}{\Delta_{12}}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \\ -\boldsymbol{\delta}\cdot\boldsymbol{t} = \frac{\Delta_{12} - \mu N_{12}}{\mu\Delta_{12}}K + \frac{\Delta_{12} - M_{12}\nu}{\nu\Delta_{12}}L' + \frac{\Delta_{12} - \Delta_{13} + \Delta_{23}}{\Delta_{12}}r. \end{aligned} \tag{6.19}$$

If we repeat our analysis for (6.6), then we find that displacement of the t_2 integrals in I_2 must occur to the right, and the poles encountered come in two series,

$$t_2^{(1)} = (K' + n_3l - n_1t_1)/n_2 > 0, \tag{6.20}$$

$$t_2^{(2)} = -\frac{m_3}{\Delta_{23}}K' + \frac{n_3}{\Delta_{23}}L - \frac{\Delta_{13}}{\Delta_{23}}t_1 > 0. \tag{6.21}$$

The $t_2^{(1)}$ sequence (6.20) gives rise to a series of contributions, the integrals in which must be displaced to the right. In doing so, two further sequences of poles,

$$t_1^{(1)} = (K' + n_2l + n_3r)/n_1,$$

$$t_1^{(2)} = \frac{-m_2}{\Delta_{12}}K' - \frac{\Delta_{23}}{\Delta_{12}}l + \frac{n_2}{\Delta_{12}}(1 + \mu r) = \frac{-m_2K' - \Delta_{23}l + n_2R}{\Delta_{12}} > 0, \tag{6.22}$$

give rise to the asymptotic series

$$I_{211} = \frac{1}{\mu n_1}\sum_{k,l,r} f_{211}(k,l,r), \quad I_{212} = \frac{1}{\Delta_{12}}\sum_{k,l,r'} f_{123}(r,k,l),$$

where

$$f_{211}(k,l,r) = \frac{(-1)^{k+l+r}}{k!l!r!}\Gamma\left(\frac{K'+n_3 l+n_2 r}{n_1}\right) \times\Gamma\left(\frac{n_1 - m_1 K' - \Delta_{13} l - \Delta_{12} r}{\mu n_1}\right)\lambda^{-\boldsymbol{\delta}\cdot\boldsymbol{t}}, \tag{6.23}$$

$$-\boldsymbol{\delta}\cdot\boldsymbol{t} = -\frac{\delta_1}{n_1}K' + \frac{\Delta_{13}-\mu N_{13}}{\mu n_1}l + \frac{\Delta_{12}-\mu N_{12}}{\mu n_1}r,$$

and where f_{123} is given in (6.19). Here, we have set $R = 1+\mu r$ (recall (2.2)).

The $t_2^{(2)}$ sequence (6.21) similarly gives rise to a series of contributions, the integrals in which must be displaced to the left. Poles that are encountered in the displacement give rise to three sequences:

$$t_1^{(1)} = -r,$$

$$t_1^{(2)} = \frac{-m_3}{\Delta_{13}}K' + \frac{n_3}{\Delta_{13}}L + \frac{\Delta_{23}}{\Delta_{13}}r \leqslant 0, \tag{6.24}$$

$$t_1^{(3)} = \frac{-m_2}{\Delta_{12}}K' + \frac{n_2}{\Delta_{12}}L - \frac{\Delta_{23}}{\Delta_{12}}r \leqslant 0. \tag{6.25}$$

These, in turn, give rise to the asymptotic series

$$I_{221} = \frac{1}{\Delta_{23}}\sum_{k,l',r'} f_{121}(l,k,r),$$

$$I_{222} = -\frac{1}{\Delta_{13}}\sum_{k,l',r'} f_{112}(l,r,k), \tag{6.26}$$

$$I_{223} = \frac{1}{\Delta_{12}}\sum_{k,l',r'} f_{123}(l,r,k),$$

where f_{121}, f_{112} and f_{123} are given, respectively, in (6.18), (6.14) and (6.19).

The triply-subscripted asymptotic series that we have obtained can be collected together in a more attractive form. In table 1, we gather together the restrictions attached to each of the formal asymptotic sums $I_{111}, I_{112}, \dots, I_{223}$; for ease of discussion, we have also displayed the denominator that appears in the factor before the Σ in each of the definitions of these sums.

The origins of the three-term inequalities present in the table arise from the restrictions in place on the final sequences of poles resulting from displacement, in order, of t_3, t_2 and t_1 contours and recorded in (6.11), (6.15), (6.16), (6.22), (6.24) and (6.25). The two-term inequalities arise at intermediate stages (after the second displacement, of t_2) from (6.7) and (6.21); the t_1 dependence there does not appear in the table since the inequality must still hold prior to the (final) displacement of the t_1 contour, when $\operatorname{Re} t_1 = 0$ everywhere except near the origin in the t_1-plane where it is positive but arbitrarily small. We also mention that series in the table sharing a common denominator for the leading factor (modulo a minus sign) also share

Table 1. *Linear restrictions imposed on the asymptotic series*
(The quantities K, K', L, L', R and R' are defined in (2.2).)

asymptotic series	denominator	inequalities
I_{111}	$m_3\nu$	none
I_{112}	Δ_{13}	$(n_3K + \Delta_{23}l - m_3R')/\Delta_{13} \leqslant 0$
I_{121}	Δ_{23}	$(n_3K - m_3L')/\Delta_{23} \leqslant 0$
I_{122}	$-\Delta_{13}$	$(n_3K - m_3L')/\Delta_{23} \leqslant 0$ $(n_3K - m_3L' + \Delta_{23}r)/\Delta_{13} \leqslant 0$
I_{123}	Δ_{12}	$(n_3K - m_3L')/\Delta_{23} \leqslant 0$ $(n_2K - m_2L' - \Delta_{23}r)/\Delta_{12} \leqslant 0$
I_{211}	μn_1	none
I_{212}	Δ_{12}	$(-m_2K' - \Delta_{23}l + n_2R)/\Delta_{12} > 0$
I_{221}	Δ_{23}	$(-m_3K' + n_3L)/\Delta_{23} > 0$
I_{222}	$-\Delta_{13}$	$(-m_3K' + n_3L)/\Delta_{23} > 0$ $(-m_3K' + n_3L + \Delta_{23}r)/\Delta_{13} \leqslant 0$
I_{223}	Δ_{12}	$(-m_3K' + n_3L)/\Delta_{23} > 0$ $(-m_2K' + n_2L - \Delta_{23}r)/\Delta_{12} \leqslant 0$

a common summand, albeit with different inequalities applying to the summand's arguments. It is this last feature that permits substantial simplification, which we illustrate in detail only for one case.

Let us examine the series with denominator $\pm\Delta_{13}$. This corresponds to the entries for I_{112}, I_{122} and I_{222}. To effect a comparison, let us make the changes of summation indices $r \to l$, $l \to r$ in I_{122} and $r \to l$, $l \to k$, $k \to r$ in I_{222}, so that all summands to the series appear as $f_{112}(k, l, r)$ (recall (6.12), (6.17), (6.26) and (6.14)). These changes of variables also effect changes to the inequalities governing I_{122} and I_{222}, rendering those for I_{122} as

$$\frac{n_3}{\Delta_{23}}K - \frac{m_3}{\Delta_{23}}R' \leqslant 0, \quad \frac{n_3}{\Delta_{13}}K + \frac{\Delta_{23}}{\Delta_{13}}l - \frac{m_3}{\Delta_{13}}R' \leqslant 0,$$

and those for I_{222} as

$$\frac{n_3}{\Delta_{23}}K - \frac{m_3}{\Delta_{23}}R' > 0, \quad \frac{n_3}{\Delta_{13}}K + \frac{\Delta_{23}}{\Delta_{13}}l - \frac{m_3}{\Delta_{13}}R' \leqslant 0.$$

The second of each of these pairs of inequalities are identical, and the first of each of these pairs are complementary. As a result, we can combine the series I_{122} and I_{222} into a single sum, which we continue to denote by I_{122}, subject to the single linear inequality

$$\frac{n_3}{\Delta_{13}}K + \frac{\Delta_{23}}{\Delta_{13}}l - \frac{m_3}{\Delta_{13}}R' \leqslant 0.$$

This inequality, however, is precisely that governing I_{112}, and since I_{112} and our 'fused' I_{122} are negatives, we see that all the series in table 1 associated with the factor Δ_{13} annihilate.

In similar fashion, we can deduce that the I_{123}, I_{212} and I_{223} series sum into a

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single series, which we continue to denote by I_{123}, subject to no linear contraints, and that the I_{121} and I_{221} series unite in another series, still denoted by I_{121}, also free of linear constraint (except, of course, that all indices be non-negative integers).

After restoring the factor $\lambda^{-1/\mu-1/\nu}$, we arrive at the large-λ asymptotic expansion of (6.4)

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}(I_{111} + I_{121} + I_{123} + I_{211}),$$

where the sums have no linear constraint, or more explicitly,

$$I(\lambda) \sim \lambda^{-1/\mu-1/\nu}\left\{\frac{1}{m_3\nu}\sum_{k,l,r} f_{111}(k,l,r) + \frac{1}{\Delta_{23}}\sum_{k,l,r} f_{121}(k,l,r) + \frac{1}{\Delta_{12}}\sum_{k,l,r} f_{123}(k,l,r) + \frac{1}{\mu n_1}\sum_{k,l,r} f_{211}(k,l,r)\right\}, \quad (6.27)$$

as $\lambda \to \infty$. The summands in (6.27) are given in (6.13), (6.18), (6.19) and (6.23), and the indices range over non-negative integers k, l and r. The principle of one asymptotic series per face of the Newton diagram continues to hold, with the first series corresponding to the face joining $(0, \nu)$ to P_3, the next corresponding to the face joining P_3 to P_2 and so on.

(*b*) *The convex case for many internal points*

A careful examination of the process used to arrive at expansions (5.22) and (6.27) shows that the result of our careful sequence of displacing integration contours is equivalent to setting various integration variables in (3.5) to the values of poles of some Γ functions appearing in the integrand of (3.5) and evaluating the remaining Γ functions in the integrand at these values. For convex cases, a high degree of regularity in this process is apparent, as we explain.

Let us set $k = N - 1$ in (3.1) and (3.5), as k will assume a different use shortly. Define the following simple determinants, paralleling the determinants used earlier in (5.2) and (6.1):

$$\Delta_{i,i+1} = \begin{vmatrix} m_i & n_i \\ m_{i+1} & n_{i+1} \end{vmatrix},$$

where $0 \leqslant i \leqslant N-1$, and where we set $P_0 = (m_0, n_0) = (\mu, 0)$ and $P_N = (m_N, n_N) = (0, \nu)$. The internal points $P_1 = (m_1, n_1), \ldots, P_{N-1} = (m_{N-1}, n_{N-1})$ are ordered so that $m_1 > m_2 > \cdots > m_{N-1}$ and $n_1 < n_2 < \cdots < n_{N-1}$. The $\Delta_{i,i+1}$ are the factors that appear in the denominators associated with series, as in (5.22) and (6.27). In order, Δ_{01} is associated with the face of the Newton diagram which connects P_0 with P_1, Δ_{12} is associated with the face joining P_1 to P_2, and so on. The asymptotic expansion of (3.1) will be of the form $\lambda^{-1/\mu-1/\nu}$ times a finite sum of series of the type given below, one series for each face in the Newton diagram.

The series associated with the face formed from P_i and P_{i+1}, $0 \leqslant i \leqslant N-1$, has the structure

$$\frac{1}{\Delta_{i,i+1}}\sum \frac{(-1)^{k+l+r_1+\cdots}}{k!l!r_1!\cdots}\Gamma(\cdot)\Gamma(\cdot)c_1^{-t_1}\cdots c_{N-1}^{-t_{N-1}}\lambda^{-\delta_1 t_1-\cdots-\delta_{N-1}t_{N-1}}, \quad (6.28)$$

with only two Γ function evaluations present in the summand. The arguments of the Γ functions are easily determined as solutions of a particular linear system, constructed from the system

$$\begin{aligned} m_1t_1+m_2t_2+\cdots+m_it_i+m_{i+1}t_{i+1}+\cdots+m_{N-1}t_{N-1} &= 1+\mu k\\ n_1t_1+n_2t_2+\cdots+n_it_i+n_{i+1}t_{i+1}+\cdots+n_{N-1}t_{N-1} &= 1+\nu l\\ t_1 \qquad\qquad\qquad\qquad\qquad\qquad &= -r_1\\ t_2 \qquad\qquad\qquad\qquad\qquad &= -r_2\\ \ddots \qquad\qquad\qquad\qquad &\cdots\\ t_{N-1} &= -r_{N-1} \end{aligned}$$

where k, l, $r_1,\dots,r_{N-1}$ are the non-negative integer parameters appearing as summation indices in the formal asymptotic sum (6.28).

For faces of the Newton diagram which do not touch the coordinate axes, namely when $0<i<N-1$, the correct linear system to use to determine the arguments of the Γ functions in (6.28) is formed by deleting the rows corresponding to the equations $t_i=-r_i$ and $t_{i+1}=-r_{i+1}$. The resulting system has only t_i and t_{i+1} as unknowns and has $\Delta_{i,i+1}$ as the determinant of the coefficient matrix

$$A=\left[\begin{array}{cccc|cc|cccc} m_1 & m_2 & \cdots & m_{i-1} & m_i & m_{i+1} & m_{i+2} & m_{i+3} & \cdots & m_{N-1}\\ n_1 & n_2 & \cdots & n_{i-1} & n_i & n_{i+1} & n_{i+2} & n_{i+3} & \cdots & n_{N-1}\\ \hline & \mathrm{I} & & & \mathrm{O} & & & \mathrm{O} & & \\ \hline & \mathrm{O} & & & \mathrm{O} & & & \mathrm{I} & & \end{array}\right],$$

where the blocks labelled I and O indicate identity and zero matrices. From Cramer's rule, t_i and t_{i+1} are given by

$$t_i=\frac{\det A_{\mathrm{col}(i)\leftarrow K^T}}{\Delta_{i,i+1}},\quad t_{i+1}=\frac{\det A_{\mathrm{col}(i+1)\leftarrow K^T}}{\Delta_{i,i+1}},$$

where $A_{\mathrm{col}(i)\leftarrow K^T}$ denotes the matrix A with ith column replaced by the transpose of $K=(1+\mu k,1+\nu l,-r_1,\dots,-r_{i-1},-r_{i+2},\dots,-r_{N-1})$. The values t_i and t_{i+1} so determined are the arguments for the Γ functions in (6.28) and together with $t_1=-r_1,t_2=-r_2,\dots,t_{i-1}=-r_{i-1},t_{i+2}=-r_{i+2},\dots,t_{N-1}=-r_{N-1}$, are the values to be used in computing the remaining factors $c_1^{-t_1}\cdots c_{N-1}^{-t_{N-1}}$ and $\lambda^{-\delta_1t_1-\cdots-\delta_{N-1}t_{N-1}}$ in the summand of (6.28). The integer parameters $k,l,r_1,\dots,r_{i-1},r_{i+2},\dots,r_{N-1}$ are the values used in the computation of the power of (-1) in the numerator and the product of factorials in the denominator of (6.28).

The series associated with the Δ_{01} and $\Delta_{N-1,N}$ faces require only a small modification of this process. For the case of the $\Delta_{N-1,N}$ series, we excise from our linear system the linear equations $\sum n_it_i=1+\nu l$ and $t_{N-1}=-r_{N-1}$ (instead of $t_i=-r_i$ and $t_{i+1}=-r_{i+1}$ for $0<i<N-1$). The solutions of this resulting system offer up t-values for use in the $c_1^{-t_1}\cdots c_{N-1}^{-t_{N-1}}$ and $\lambda^{-\delta_1t_1-\cdots-\delta_{N-1}t_{N-1}}$ factors in the summand of (6.28). The arguments of the Γ functions in (6.28) are the computed t_{N-1} and $(1-\sum n_it_i)/\nu$.

For the Δ_{01} series, the appropriate pair of linear equations to remove is $t_1=-r_1$ and $\sum m_it_i=1+\mu k$. The Γ functions in (6.28) are evaluated at the computed t_1 and $(1-\sum m_it_i)/\mu$.

Leading terms for the series associated with each face of the Newton diagram are easily obtained. For the face joining P_0 to P_1, the leading term is

$$\frac{1}{\mu n_1}\Gamma\left(\frac{1}{n_1}\right)\Gamma\left(\frac{n_1-m_1}{\mu n_1}\right)c_1^{-1/n_1}\lambda^{-\delta_1/n_1},$$

where use has been made of $\Delta_{01} = \mu n_1$. For a face not touching the coordinate axes, $\overline{P_i P_{i+1}}$ with $0 < i < N-1$, the leading term is

$$\frac{1}{\Delta'}\Gamma\left(\frac{n_{i+1}-m_{i+1}}{\Delta'}\right)\Gamma\left(\frac{m_i-n_i}{\Delta'}\right) \times c_i^{(m_{i+1}-n_{i+1})/\Delta'} c_{i+1}^{(n_i-m_i)/\Delta'} \lambda^{-(\nu M_{i,i+1}-\Delta')/\nu\Delta'-(\mu N_{i,i+1}-\Delta')/\mu\Delta'},$$

where we have used the identities (6.1) and (6.2), extended for the case of several internal points, and written Δ' for $\Delta_{i,i+1}$. Finally, the face joining P_{N-1} to P_N has the leading term

$$\frac{1}{m_{N-1}\nu}\Gamma\left(\frac{1}{m_{N-1}}\right)\Gamma\left(\frac{m_{N-1}-n_{N-1}}{m_{N-1}\nu}\right) c_{N-1}^{-1/m_{N-1}}\lambda^{-\delta_{N-1}/m_{N-1}},$$

where use has been made of $\Delta_{N-1,N} = m_{N-1}\nu$.

With these leading terms, it is a simple matter to conduct an analysis of the relationship between remoteness of the Newton diagram and the order of the leading term in the asymptotic expansion of $I(\lambda)$ in this case. If we denote by m_i^* the abscissa of the point of intersection of the line through the ith face (formed by P_{i-1} and P_i) with the diagonal $m = n$, then it is routine to show that the quantities m_i^* increase until the face meeting the diagonal is met, and then decrease again. If this face is internal (i.e. one that does not meet the coordinate axes), then P_{i-1} and P_i lie on opposite sides of the diagonal, in which case $n_i - m_i$ and $m_{i-1} - n_{i-1}$ are both positive and the leading term corresponding to this i then has Γ functions with positive arguments. For that i, it is then a straightforward matter to show that the remoteness $-1/m_i^*$ equals

$$-\frac{1}{\mu}-\frac{1}{\nu}-\frac{\nu M_{i-1,i}-\Delta_{i-1,i}}{\nu\Delta_{i-1,i}}-\frac{\mu N_{i-1,i}-\Delta_{i-1,i}}{\mu\Delta_{i-1,i}}.$$

A similar argument can be advanced for the two remaining cases (when the diagonal does not meet an internal face). The details for these arguments are left to the reader.

7. Numerical examples

In this section we present numerical examples to illustrate the accuracy of the various expansions of $I(\lambda)$ developed in §§ 3–6. These expansions were developed on the assumption that all the poles in the corresponding Mellin–Barnes integrals were simple. Here we present cases where this condition is satisfied but also show how to deal with situations where double poles are present which give rise to logarithmic terms.

Example 7.1. We consider the integral with a single internal point given by

$$I(\lambda) = \int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu + x^{m_1}y^{n_1} + y^\nu)]\,\mathrm{d}x\mathrm{d}y, \tag{7.1}$$

where we assume that $\delta_1 = 1 - m_1/\mu - n_1/\nu > 0$, i.e. the vertex $P_1 = (m_1, n_1)$ lies in front of the back face of the Newton diagram joining the points $(\mu, 0)$ and $(0, \nu)$. Without loss of generality we have put the constant c_1 multiplying the term $x^{m_1}y^{n_1}$ equal to unity, since a simple scaling of the variables x, y and the parameter

λ enables the general case to be expressed in the above form. From (4.1) this has the Mellin representation

$$I(\lambda)=\frac{\lambda^{-1/\mu-1/\nu}}{\mu\nu}\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\Gamma(t)\Gamma\left(\frac{1-m_1t}{\mu}\right)\Gamma\left(\frac{1-n_1t}{\nu}\right)\lambda^{-\delta_1 t}\,\mathrm{d}t, \tag{7.2}$$

valid in the sector $|\arg\lambda|<\frac{1}{2}\pi(1+m_1/\mu+n_1/\nu)/\delta_1$ (see (3.7)).

When all the poles of the integrand in (7.2) are simple, the asymptotic expansion of $I(\lambda)$, given by (4.7), consists of two series with the asymptotic scales $\lambda^{-\delta_1\mu/m_1}$ and $\lambda^{-\delta_1\nu/n_1}$. For example, in the case $\mu=\nu=3$ and $m_1=\frac{3}{2}$, $n_1=1$, we obtain the expansion

$$I(\lambda)\sim\tfrac{2}{9}\lambda^{-7/9}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(\tfrac{2}{3}+2k)\Gamma(\tfrac{1}{9}-\tfrac{2}{3}k)\lambda^{-k/3} +\tfrac{1}{3}\lambda^{-5/6}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(1+3k)\Gamma(-\tfrac{1}{6}-\tfrac{3}{2}k)\lambda^{-k/2},$$

associated with the scales $\lambda^{-1/3}$ and $\lambda^{-1/2}$, respectively. In table 2 we present the results of numerical calculations for different values of the parameters μ,ν,m_1 and n_1. The second column shows the value of $I(\lambda)$ calculated either from (7.1) by standard numerical quadrature or from its equivalent representation as a generalized hypergeometric function given by the right-hand side of (4.8). The third column shows the asymptotic value of $I(\lambda)$ obtained by optimal truncation of each asymptotic series, that is, truncation just before the numerically smallest term.

An example of an integral of type (7.1) involving double poles is given by $\mu=\nu=5$ and $m_1=n_1=2$ for which the expansion (4.7) becomes nugatory. In this case the integrand in (7.2) possesses a sequence of double poles at $t=\frac{1}{2}+\frac{5}{2}k$, $k=0,1,2,\ldots$. The residues at these poles are given by the coefficient of x^{-1} in the Maclaurin expansion of

$$\begin{aligned}\Gamma(\tfrac{1}{2}+\tfrac{5}{2}k+x)&\Gamma^2(-k-\tfrac{2}{5}x)\lambda^{-k/2-1/10-x/5}\\ &=\frac{\lambda^{-k/2-1/10}\Gamma(\frac{1}{2}+\frac{5}{2}k)}{(2x/5)^2(k!)^2}\frac{[1+\psi(\frac{1}{2}+\frac{5}{2}k)x+\cdots][1-\frac{1}{5}x\log\lambda+\cdots]}{[1+\frac{2}{5}\psi(1+k)x+\cdots]^2}\\ &=\frac{25}{4}\lambda^{-k/2-1/10}\frac{\Gamma(\frac{1}{2}+\frac{5}{2}k)}{x^2(k!)^2}\{1+x[\psi(\tfrac{1}{2}+\tfrac{5}{2}k)-\tfrac{1}{5}\log\lambda-\tfrac{4}{5}\psi(1+k)]+\cdots\},\end{aligned}$$

where ψ denotes the logarithmic derivative of the Γ function. The asymptotic expansion of $I(\lambda)$ in this case therefore consists of the single expansion

$$I(\lambda)\sim\frac{1}{20}\lambda^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\frac{5}{2}k)}{(k!)^2}\{\log\lambda+4\psi(1+k)-5\psi(\tfrac{1}{2}+\tfrac{5}{2}k)\}\lambda^{-k/2},$$

as $\lambda\to\infty$.

It can be observed in the examples chosen that there is a considerable variation in the asymptotic scales according to the proximity of the vertex P_1 to the back face (see § 5 *f*). This results in a rather wide range of λ-values necessary to achieve an adequate description of the behaviour of $I(\lambda)$ by the corresponding asymptotic formula. In the symmetrical case with $\mu=\nu$ and $m_1=n_1$, all the poles in the integrand in (7.2) are double and each term in the expansion involves a term in

Table 2. *Comparison of the asymptotic values of $I(\lambda)$ with one internal point*

λ	$I(\lambda)$	asymptotic value
$\mu=\nu=3,\ m_1=\frac{3}{2},\ n_1=1,\ \delta_1=\frac{1}{6}$		
1.0×10^2	0.02643 82192	0.01952 49523
1.0×10^3	0.00511 42751	0.00507 50847
5.0×10^3	0.00160 24135	0.00160 24277
1.0×10^4	0.00096 91553	0.00096 91550
5.0×10^4	0.00029 95087	0.00029 95086
$\mu=\nu=4,\ m_1=\frac{1}{2},\ n_1=\frac{4}{5},\ \delta_1=\frac{27}{40}$		
1.0×10^1	0.07981 13640	0.05396 33427
2.0×10^1	0.03703 09129	0.03707 77826
5.0×10^1	0.01235 59580	0.01235 59563
8.0×10^1	0.00686 99475	0.00686 99466
1.0×10^2	0.00517 85557	0.00517 85557
$\mu=\nu=5,\ m_1=n_1=2,\ \delta_1=\frac{1}{5}$		
1.0×10^2	0.11047 54348	0.10919 62332
5.0×10^2	0.05523 15228	0.05520 87617
1.0×10^3	0.04087 48889	0.04087 42206
1.0×10^4	0.01488 32370	0.01488 32334
5.0×10^4	0.00728 03604	0.00728 03598

$\log\lambda$. If we relax the degree of symmetry of the Newton diagram by taking $\mu\neq\nu$ (but with m_1, n_1 still situated on the 45° line), it is then easily seen (when μ/ν is rational) that both single and double poles can arise, with the first pole of the sequence always being double. This means that, although the leading term again contains $\log\lambda$, the other logarithmic contributions now appear as higher order terms distributed amongst the algebraic terms in the expansion. When there is no longer any symmetry in the Newton diagram (i.e. when $\mu\neq\nu$ and $m_1\neq n_1$), the poles can either be all simple (as in the first two examples in table 2) or there can be double poles embedded within the sequence of single poles. In either case, however, the leading term of the expansion in the unsymmetric case is always algebraic.

Finally, as mentioned in § 3, the expansion (4.7) (and the corresponding expansion involving $\log\lambda$ in the case of double poles) is valid for complex λ in a sector enclosing the rays $\arg\lambda=\pm\frac{1}{2}\pi$. To illustrate this fact we show in table 3 the values of $I(\lambda)$ when $\mu=\nu=5$ and $m_1=1$, $n_1=\frac{1}{2}$ computed from the right-hand side of (4.8) for varying phase of λ with fixed $|\lambda|$. In this case the sector of validity of the integral (7.2), and hence that of the expansion (4.7) (in the sense of Poincaré), is $|\arg\lambda|<\frac{13}{14}\pi$. We remark that $\arg\lambda=\pm\frac{1}{2}\pi$ corresponds to a Fourier integral with a single critical point at the origin.

Table 3. *Comparison of the asymptotic values of $I(\lambda)$ with a single internal point for complex λ*

θ/π	$\mu=\nu=5,\ m_1=1,\ n_1=\frac{1}{2},\ \delta_1=\frac{7}{10},\ \lambda=50e^{i\theta}$ $I(\lambda)$	asymptotic value
0.0	0.01578 04781 + 0.00000 00000i	0.01578 04780 + 0.00000 00000i
0.1	0.01481 90028 − 0.00553 01069i	0.01481 90025 − 0.00553 01067i
0.2	0.01202 10294 − 0.01044 67200i	0.01202 10301 − 0.01446 72144i
0.3	0.00764 78949 − 0.01417 01485i	0.00764 78957 − 0.01417 01492i
0.4	0.00213 95833 − 0.01619 66016i	0.00213 95841 − 0.01619 65996i
0.5	−0.00388 77161 − 0.01615 67121i	−0.00388 77151 − 0.01615 67144i

Example 7.2. As an example of an integral with two internal points, we consider

$$I(\lambda)=\int_0^\infty\int_0^\infty \exp[-\lambda(x^\mu+c_1x^{m_1}y^{n_1}+c_2x^{m_2}y^{n_2}+y^\nu)]\,dx dy,$$

where the parameters will be chosen to correspond to a convex Newton diagram. From (5.1), $I(\lambda)$ has the Mellin representation

$$\frac{\lambda^{-1/\mu-1/\nu}}{\mu\nu}\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right)^2\Gamma(t_1)\Gamma(t_2)\Gamma\left(\frac{1-m_1t_1-m_2t_2}{\mu}\right)\Gamma\left(\frac{1-n_1t_1-n_2t_2}{\nu}\right) \times c_1^{-t_1}c_2^{-t_2}\lambda^{-\delta_1t_1-\delta_2t_2}\,dt_1dt_2, \tag{7.3}$$

defined in the sector (see (3.7))

$$|\arg\lambda|<\min_{i=1,2}\{\tfrac{1}{2}\pi(1+m_i/\mu+n_i/\nu)/\delta_i\},$$

when c_1, c_2 are assumed to be positive real.

When all the poles of the integrand in (7.3) are simple, the asymptotic expansion of $I(\lambda)$ consists of three series given by (5.22) in the case $c_1=c_2=1$. An illustration of such a situation is $\mu=5$, $\nu=4$ with $(m_1,n_1)=(\frac{5}{2},1)$ and $(m_2,n_2)=(1,2)$. In this case we have $\delta_1=\frac{1}{4}$, $\delta_2=\frac{3}{10}$, $\Delta=4$ and $\Delta-\mu N=-1$, $\Delta-\nu M=-2$ so that, from (5.3), the associated Newton diagram is convex. The expansion of $I(\lambda)$ then takes the form

$$I(\lambda)\sim\lambda^{-9/20}\left\{\frac{1}{4}\sum_{k,l}f_{11}(k,l)+\frac{1}{4}\sum_{k,l}f_{12}(k,l)+\frac{1}{5}\sum_{k,l}f_{21}(k,l)\right\}, \tag{7.4}$$

as $\lambda\to\infty$, where $k,l=0,1,2,\ldots$, and, from (5.11), (5.12), (5.14), (5.15), (5.19) and (5.20),

$$f_{11}(k,l)=\frac{(-1)^{k+l}}{k!l!}\Gamma(1+5k+\tfrac{5}{2}l)\Gamma(-\tfrac{1}{4}-\tfrac{5}{2}k-l)\lambda^{-3/10-3k/2-l/2},$$

$$f_{12}(k,l)=\frac{(-1)^{k+l}}{k!l!}\Gamma(\tfrac{1}{4}+\tfrac{5}{2}k-l)\Gamma(\tfrac{3}{8}-\tfrac{5}{4}k+\tfrac{5}{2}l)\lambda^{-7/40-k/4-l/2},$$

$$f_{21}(k,l)=\frac{(-1)^{k+l}}{k!l!}\Gamma(1+4k+2l)\Gamma(-\tfrac{3}{10}-2k-\tfrac{4}{5}l)\lambda^{-1/4-k-l/5}.$$

When $c_1,c_2\neq1$, additional terms involving powers of c_1 and c_2 appear in each of

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Table 4. *Comparison of the asymptotic values of $I(\lambda)$ with two internal points*

λ	$I(\lambda)$	asymptotic value
$\mu=5$, $\nu=4$, $(m_1,n_1)=(2.5,1)$, $(m_2,n_2)=(1,2)$, $c_1=c_2=1$		
1.0×10^3	0.01649 86855	0.01530 34083
5.0×10^3	0.00660 18949	0.00662 55882
1.0×10^4	0.00442 75336	0.00442 22631
1.0×10^5	0.00115 31511	0.00115 33535
5.0×10^5	0.00044 41628	0.00044 40313
1.0×10^6	0.00029 36705	0.00029 36729
$\mu=5$, $\nu=4$, $(m_1,n_1)=(2.5,1)$, $(m_2,n_2)=(1,2)$, $c_1=2$, $c_2=3$		
1.0×10^3	0.01078 82133	0.01228 82165
5.0×10^3	0.00418 68491	0.00421 90510
1.0×10^4	0.00277 62231	0.00278 25077
1.0×10^5	0.00070 10746	0.00070 10886
5.0×10^5	0.00026 56345	0.00026 56349
1.0×10^6	0.00017 45786	0.00017 45787
$\mu=\nu=4$, $(m_1,n_1)=(2,1)$, $(m_2,n_2)=(1,2)$, $c_1=c_2=1$		
1.0×10^3	0.01185 73873	0.01173 42624
5.0×10^3	0.00447 05490	0.00447 33081
1.0×10^4	0.00292 42695	0.00292 60266
1.0×10^5	0.00070 23219	0.00070 23687
5.0×10^5	0.00025 58167	0.00025 58160
1.0×10^6	0.00016 51223	0.00016 51222

the expansions in (7.4). From § 5, the summands in (7.4) are then replaced by

$$f_{11}(k,l)c_1^l c_2^{-(1+5k+5l/2)},$$
$$f_{12}(k,l)c_1^{-(1/4+5k/2-l)}c_2^{-(3/8+5k/4-5l/2)},$$
$$f_{21}(k,l)c_1^{-(1+4k+2l)}c_2^l,$$

respectively. In table 4 we present the results of numerical calculations of $I(\lambda)$ when $c_1=c_2=1$ and $c_1=2$, $c_2=3$.

The second case we examine has $\mu=\nu=4$, $c_1=c_2=1$ with $(m_1,n_1)=(2,1)$ and $(m_2,n_2)=(1,2)$, so that the internal points are again in front of the back face but now straddle the 45° line in a symmetrical manner. Not unexpectedly, it will be found that the high degree of symmetry in the Newton diagram in this case will involve double poles which generate terms in $\log\lambda$. We follow the procedure outlined in § 5 where the t_2 contour in (7.3) is displaced to the right to yield the result

$$I(\lambda)=\lambda^{-1/2}(I_1+I_2), \tag{7.5}$$

where, from (5.6) and (5.7),

$$I_1 = \frac{1}{4}\sum_k \frac{(-1)^k}{k!}\lambda^{-k-1/4}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1)\Gamma(1+4k-2t_1)\Gamma(-\tfrac{1}{4}-2k+\tfrac{3}{4}t_1)\lambda^{t_1/4}\,\mathrm{d}t_1,$$

$$I_2 = \frac{1}{8}\sum_k \frac{(-1)^k}{k!}\lambda^{-k/2-1/8}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \Gamma(t_1)\Gamma(\tfrac{1}{2}+2k-\tfrac{1}{2}t_1)\Gamma(\tfrac{1}{8}-\tfrac{1}{2}k-\tfrac{3}{8}t_1)\lambda^{-t_1/8}\,\mathrm{d}t_1.$$

As shown at (5.8), the t_1 contour in each integral in I_1 must be displaced to the left. We thus encounter two sequences of poles at $t_1^{(1)} = 0, -1, -2, \ldots$ and $t_1^{(2)} = \frac{1}{3} + \frac{8}{3}k - \frac{4}{3}l$, $l = 0, 1, 2, \ldots$, where the latter sequence is subject to the constraint $4l \geqslant 8k+1$ to ensure that $t_1^{(2)} \leqslant 0$. When $l = 1+2k+3n$, $n = 0, 1, 2, \ldots$, the corresponding poles in the $t_1^{(2)}$ sequence coincide with poles of the $t_1^{(1)}$ sequence to form double poles at $t = -1-4n$ with residue

$$\tfrac{4}{3}(-1)^n \frac{\Gamma(3+4k+8n)}{\Gamma(2+4n)\Gamma(2+2k+3n)}\lambda^{-k-n-1/2} \times\{\tfrac{1}{4}\log\lambda - 2\psi(3+4k+8n) + \psi(2+4n) + \tfrac{3}{4}\psi(2+2k+3n)\}. \quad (7.6)$$

Evaluation of the contribution from the simple poles is as described in § 5 and we consequently find that for large λ

$$I_1 \sim \tfrac{1}{4}\lambda^{-1/4}\sum_{\substack{k=0\\ l\neq 1+4n}}^{\infty}\sum_{l=0}^{\infty} f_{11}(k,l) + \tfrac{1}{3}\lambda^{-1/6}\sum_{\substack{k=0\\ l\neq 1+2k+3n\\ 4l\geqslant 8k+1}}^{\infty}\sum_{l=0}^{\infty} f_{12}(k,l) + \tfrac{1}{3}\lambda^{-1/2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g(k,l),$$

where n is a non-negative integer and, from (5.11), (5.12), (5.14) and (5.15),

$$f_{11}(k,l) = \frac{(-1)^{k+l}}{k!l!}\Gamma(1+4k+2l)\Gamma(-\tfrac{1}{4}-2k-\tfrac{3}{4}l)\lambda^{-k-l/4},$$

$$f_{12}(k,l) = \frac{(-1)^{k+l}}{k!l!}\Gamma(\tfrac{1}{3}+\tfrac{8}{3}k-\tfrac{4}{3}l)\Gamma(\tfrac{1}{3}-\tfrac{4}{3}k+\tfrac{8}{3}l)\lambda^{-k/3-l/3}$$

and

$$g(k,l) = \frac{(-1)^{k+l}}{k!}\frac{\Gamma(3+4k+8l)}{\Gamma(2+4l)\Gamma(2+2k+3l)}\lambda^{-k-l} \times\{\tfrac{1}{4}\log\lambda - 2\psi(3+4k+8l) + \psi(2+4l) + \tfrac{3}{4}\psi(2+2k+3l)\}.$$

Proceeding in a similar fashion for I_2 upon displacement of the t_1 contour to the right, we encounter two sequences of poles at $t_1^{(1)} = 1+4k+2l$ and $t_1^{(2)} = \frac{1}{3} - \frac{4}{3}k + \frac{8}{3}l$, $l = 0, 1, 2, \ldots$, where the latter sequence is subject to the constraint $4k < 8l+1$ to ensure that $t_1^{(2)} > 0$. Double poles arise at $t = 3+4k+8n$, $n = 0, 1, 2, \ldots$, corresponding to the values of l in $t_1^{(2)}$ given by $l = 1+2k+3n$, with residues equal to a factor of -2 times the value in (7.6). Hence, as $\lambda \to \infty$, we find

$$I_2 \sim \tfrac{1}{4}\lambda^{-1/4}\sum_{\substack{k=0\\ l\neq 1+4n}}^{\infty}\sum_{l=0}^{\infty} f_{11}(k,l) + \tfrac{1}{3}\lambda^{-1/6}\sum_{\substack{k=0\\ l\neq 1+2k+3n\\ 4k<8l+1}}^{\infty}\sum_{l=0}^{\infty} f_{12}(l,k) + \tfrac{1}{3}\lambda^{-1/2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g(k,l).$$

Collecting together I_1 and I_2 according to (7.5), we find that the series involving

$\lambda^{-1/4}$ and $\lambda^{-1/2}$ are equal while those involving $\lambda^{-1/6}$ yield

$$\tfrac{1}{3}\lambda^{-1/6}\Bigg\{\sum_{\substack{k=0\\ l\neq 1+2k+3n\\ 4l\geqslant 8k+1}}^{\infty}\sum_{l=0}^{\infty} f_{12}(k,l)+\sum_{\substack{k=0\\ l\neq 1+2k+3n\\ 4k<8l+1}}^{\infty}\sum_{l=0}^{\infty} f_{12}(l,k)\Bigg\}.$$

If we relabel the summation indices by putting $k\to l$ and $l\to k$ in the second expansion, we see that the constraint $4l\geqslant 8k+1$ on the first expansion is exactly the complement of the constraint $4l<8k+1$ on the second expansion. Consequently, the two expansions can be expressed as a single expansion subject only to the constraints $k\neq 1+2l+3n$ and $l\neq 1+2k+3n$.

Hence, we finally obtain the expansion of $I(\lambda)$ in the form (for $n=0,1,2,\ldots$)

$$I(\lambda)\sim \tfrac{1}{2}\lambda^{-3/4}\sum_{\substack{k=0\\ l\neq 1+4n}}^{\infty}\sum_{l=0}^{\infty} f_{11}(k,l)+\tfrac{1}{3}\lambda^{-2/3}\sum_{\substack{k=0\\ k\neq 1+2l+3n\\ l\neq 1+2k+3n}}^{\infty}\sum_{l=0}^{\infty} f_{12}(k,l)+\tfrac{2}{3}\lambda^{-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g(k,l), \tag{7.7}$$

as $\lambda\to\infty$. We note that the restrictions appearing on the first two expansions in (7.7) simply correspond to the deletion from these sums of the singular terms resulting from the double poles. The numerical values of $I(\lambda)$ in this case are shown in table 4 and are compared with the asymptotic values computed from (7.7).

8. Summary

The analysis provided in this paper illustrates a number of principles, foremost of which is that the method of representation of Laplace-type integrals as iterated Mellin–Barnes integrals, followed by conventional residue computations, is easily applied, produces complete asymptotic expansions and can be readily carried out without regard to the geometry of the associated Newton diagram. This method is also applicable to a class of Fourier integrals, as illustrated in § 7. This is a consequence of the wide range of values of $|\arg\lambda|$ (described in § 3), of at least $\frac{1}{2}\pi$ for positive real constants $c_1,\ldots,c_k$ (recall (3.1)), for which the expansions are valid in the Poincaré sense.

The dimensionality of the iterated Mellin–Barnes integral representation (3.5) of $I(\lambda)$ is governed by the number of terms in the phase to which (3.4) is applied, and not by the dimensionality of the original Laplace integral representation (3.1). Accordingly, even one-dimensional Laplace-type integrals can lead to high dimension iterated Mellin–Barnes integrals. An investigation of one-dimensional integrals, for which the geometry of the Newton diagram of the phase is trivial, is undertaken in Kaminski & Paris (1997). There the reader will find an account of the asymptotic nature of the expansions arising from the method of representation as Mellin–Barnes integrals, a concern not presented in this paper for reasons of length.

The retention of geometric information present in the Newton diagram, however, is a great deal more insightful. It can be seen that the asymptotic expansion of $I(\lambda)$ (recall (3.1)) is a compound expansion in which each constituent asymptotic series is associated with a single face of the Newton diagram of the phase function. Terms of the phase function not corresponding to vertices of the Newton diagram do not generate asymptotic series, and their presence appears in the expansion of $I(\lambda)$ only in the arguments of the Γ functions found in expansions associated with faces of the Newton diagram. Furthermore, these terms (in the language of § 5, such terms correspond to points behind the Newton diagram) play no role in the leading terms of the series comprising the asymptotic expansion of $I(\lambda)$. This makes the study of the relationship between 'remoteness' of the Newton diagram and the order of the leading term of the expansion of $I(\lambda)$ particularly simple to undertake, as points behind the Newton diagram can safely be ignored.

The correspondence between individual series in the compound expansion of $I(\lambda)$ and the faces of the Newton diagram only fails in cases of extreme symmetry with respect to the line $m = n$ in the plane of the diagram. In such cases, illustrated in § 7, the compound expansion degenerates into one with fewer component series, some of which have a high frequency of logarithmic terms. Additionally, for less symmetric Newton diagrams where the correspondence does not break down, it is apparent that greater symmetry manifests itself in the expansions by yielding more logarithmic terms in the constituent series.

The geometric interpretation of the various quantities constituting the asymptotic scales for the expansions developed in this paper, detailed for the case of two internal points in the convex case in § 5 f, can be readily extended to cases with more internal points. Such a geometric approach can even be extended to deal with treble Laplace-type integrals, with little modification provided one examines volumes of tetrahedra instead of areas of triangles, as was undertaken in § 5 f.

The numerical investigations of § 7 reveal a number of interesting issues involving the practical application of the expansions obtained in the paper. One is that of the role played by the quantities δ_i (cf. (3.2)). While possessing a simple geometric interpretation, the quantities δ_i serve to highlight the worsening utility of the asymptotic expansions presented in this paper for small values of δ_i. In such small-δ_i cases, other approximation techniques may prove more efficacious when the large parameter λ is of modest magnitude.

Another concern raised by the investigations of § 7 arises frequently in cases when the parameters μ, ν, m_i and n_i result in arguments of the Γ functions in the compound expansions (e.g. the quantities f_{ij} to be found in (5.11), (5.19) and elsewhere) approaching poles of those Γ functions. In such cases, the quality of the numerical approximations furnished by the asymptotic approximations deteriorates, although the effect of 'near misses' of poles of Γ functions might manifest itself only in comparatively late terms of the asymptotic expansions.

We note that this study of double Laplace-type integrals also provides a rich setting in which to investigate a host of uniformity problems. Two easily-framed examples of uniformity problems would be: (i) to analyse the transition in the expansions when a vertex of the Newton diagram is displaced away from the origin into the region behind the diagram; and (ii) to analyse the changes in the expansions as a constant, c_i (recall (3.1)), tends to zero. In situation (i), a smooth deformation of the phase function is taking place, but at some point of the deformation process the Newton diagram will have two adjacent faces fuse into a single face with a corresponding reduction in the number of series in the compound asymptotic expansion of $I(\lambda)$. In case (ii), likely to be of greater practical concern, the compound expansion has the same number of series for all $c_i > 0$, but abruptly loses a series when c_i vanishes.

Finally, we close by making a few remarks on the extension of the results in this paper to the more general integral (1.1) when $n = 2$, with analytic phase f (restricted so as to have no saddle points in the domain of integration) and amplitude g. If we write $f(x, y) = f_0(x, y) + f_1(x, y)$, where f_0 comprises those terms in the Maclaurin expansion of f that correspond to vertices of the Newton diagram of f, then we can expand the resulting factor, $\exp[-\lambda f_1(x, y)]g(x, y)$, appearing in the integrand into its Maclaurin series $\sum_{r,s}(-\lambda)^{\nu(r,s)}p_{r,s}(\lambda)x^r y^{\mathrm{s}}$, where r, s and $\nu(r, s)$ are non-negative integers and $p_{r,s}(\lambda)$ is a polynomial in inverse powers of λ. This then leads to the formal series expansion

$$I(\lambda) = \sum_{r,s}(-\lambda)^{\nu(r,s)}p_{r,s}(\lambda) \cdot I_{r,s}(\lambda), \tag{8.1}$$

where

$$I_{r,s}(\lambda) = \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda f_0(x,y)} x^r y^{\mathrm{s}} \, \mathrm{d}x\mathrm{d}y.$$

The integrals $I_{r,s}(\lambda)$ are amenable to our method of evaluation (see the remark at the end of § 3) and produce asymptotic expansions, which enjoy the same geometric connections as $I_{0,0}(\lambda)$, given by the compound series

$$I_{r,s}(\lambda) \sim \lambda^{-\eta(r,s)} \sum_{j=1}^{N} \sum_{\alpha} a_{r,s,j}(\alpha)\lambda^{-\zeta_j(\alpha)}, \quad \lambda \to \infty. \tag{8.2}$$

The outer sum represents the sum of component series in the compound expansion (one series for each of the N faces of the Newton diagram). To each of these faces is associated an asymptotic scale $\lambda^{-\zeta_j(\alpha)}$, with α a vector of non-negative integers of dimension equal to the number of internal vertices of the Newton diagram. Combination of (8.1) and (8.2) then, in principle, supplies the asymptotic expansion of $I(\lambda)$ as $\lambda \to \infty$ in the more general case. The structure of this result, however, is seen to be rather unwieldy, and this is compounded by the fact that the determination of the terms $p_{r,s}(\lambda)$ and the powers $\nu(r, s)$ is an analytically difficult process. In addition, for treble (and higher dimensional) integrals treated in the sequel, this problem is likely to be even more acute as a result of the need to accommodate the possibility of non-triangular faces in the Newton diagram.

The themes of the investigations carried out in this paper for double Laplace-type integrals are pursued in a sequel for treble integrals, where the richer geometric structure of the Newton diagram and its relation to the method utilized here are explored.

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References

Arnold, V. I., Gussein-Zade, S. M. & Varchenko, A. N. 1988 *Singularities of differentiable maps II*. Berlin: Birkhäuser.

Brieskorn, E. & Knörrer, H. 1986 *Plane algebraic curves.* Basel: Birkhäuser.

Dostal, M. & Gaveau, B. 1987 Développements asymptotiques explicites d'intégrales de Fourier pour certains points critiques dégénérés. *C. R. Acad. Sci. Paris* I **305**, 857–859.

Dostal, M. & Gaveau, B. 1989 The stationary phase method for certain degenerate critical points. I. *Can. J. Math.* **41**, 907–931.

Denef, J. & Sargos, P. 1989 Polyèdre de Newton et distribution f_+^{s}. I. *J. d'Anal. Math.* **53**, 201–218.

Jones, D. S. & Kline, M. 1958 Asymptotic expansions of multiple integrals and the method of stationary phase. *J. Math. Phys.* **37**, 1–28.

Kaminski, D. & Paris, R. B. 1997 Asymptotics via iterated Mellin–Barnes integrals: application to the generalised Faxén integral. *Methods Appl. Anal.* **4**, 311–325.

Malgrange, B. 1974 Intégrales asymptotiques et monodromie. *Ann. Sci. Éc. Norm. Sup.* IV **7**, 405–430.

Olver, F. W. J. 1974 *Asymptotics and special functions.* London: Academic.

Paris, R. B. & Wood, A. D. 1986 *Asymptotics of high order differential equations*, Pitman Res. Notes in Math. Series, vol. 129. Harlow: Longman Scientific & Technical.

Slater, L. J. 1966 *Generalised hypergeometric functions.* Cambridge University Press.

Vasil'ev, V. A. 1977 Asymptotic exponential integrals, Newton's diagram, and the classification of minimal points. *Functional Analyt. Appl.* **11**, 163–172.

Wong, R. 1989 *Asymptotic expansions of integrals.* London: Academic.

Wong, R. & McClure, J. P. 1981 On a method of asymptotic evaluation of multiple integrals. *Math. Comp.* **37**, 509–521.

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